Stability Analysis of RCP for Long-lived Flows

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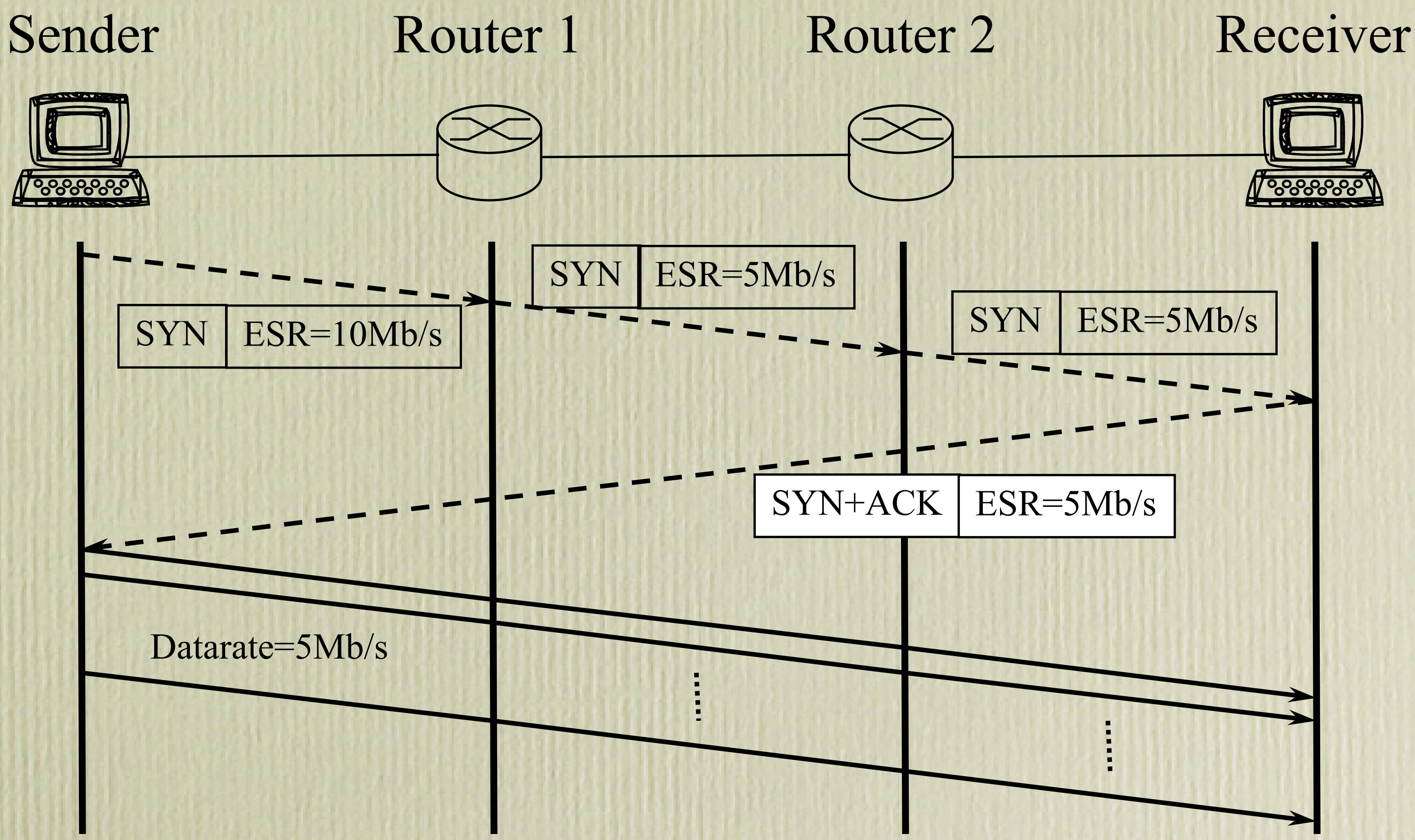
Rate Control Protocol (RCP)



RCP Equation

$$R(t) = R(t - d_0) + \frac{\alpha(C - y(t)) - \beta\frac{q(t)}{d(t)}}{N}$$

$$d(t) = d_0 + \frac{q(t)}{C}$$

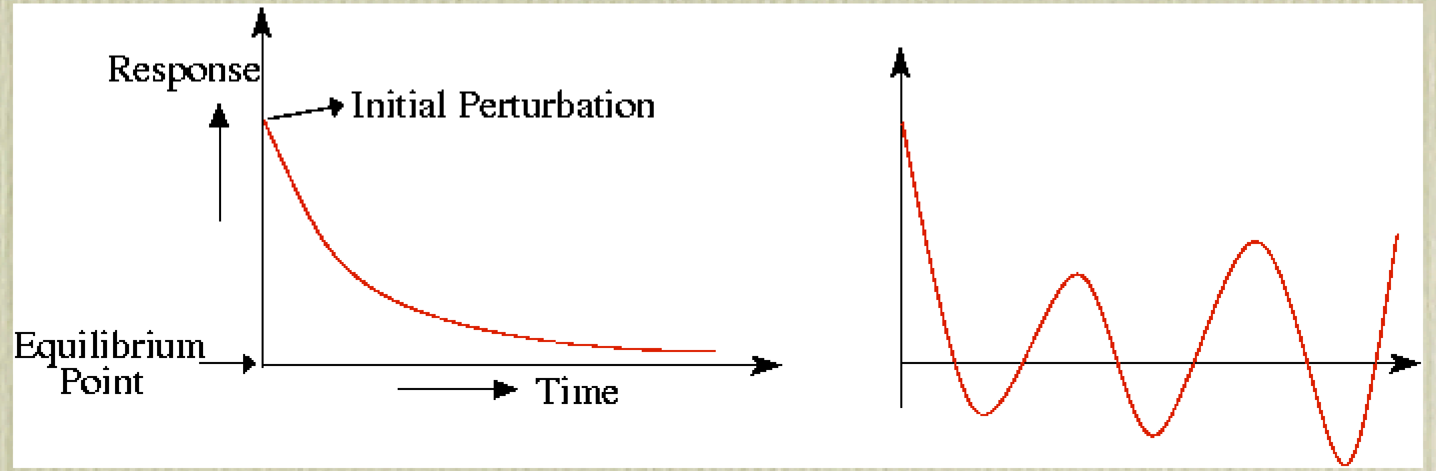
$$N = \frac{C}{R(t - d)}$$

$$R(t) = R(t - d_0)[1 + \frac{\alpha(C - y(t)) - \beta\frac{q(t)}{d(t)}}{C}]$$

$$R(t) = R(t - T)[1 + \frac{\frac{T}{d(t)}[\alpha(C - y(t)) - \beta\frac{q(t)}{d(t)}]}{C}]$$

Stability

What is a stable system?



Goal of RCP Stability Analysis

- determine $\alpha, \ \beta$
- dependence on N, C, d

General Methodology of Analysis

- System description
- Find equilibrium point
- Linearize around equilibrium point
- Transfer function of the system
- Frequency response methods
  - Bode plots
  - Nyquist criterion

RCP System Equations

$$\dot{R}(t) = R(t-T)\left[\frac{\alpha(C - y(t)) - \beta\frac{q(t)}{d(t)}}{Cd(t)}\right]$$

$$d(t) = d_0 + \frac{q(t)}{C}$$

$$\dot{q}(t) = \begin{cases} [y(t) - C] \text{ if } q(t) > 0 \\ [y(t) - C]^+ \text{ if } q(t) = 0 \end{cases}$$

$$y(t) = N \times R(t-d)$$

Equilibrium Point and Linearization

Equilibrium

$$\dot{R}(t) = 0; \;\; \dot{q}(t) = 0$$

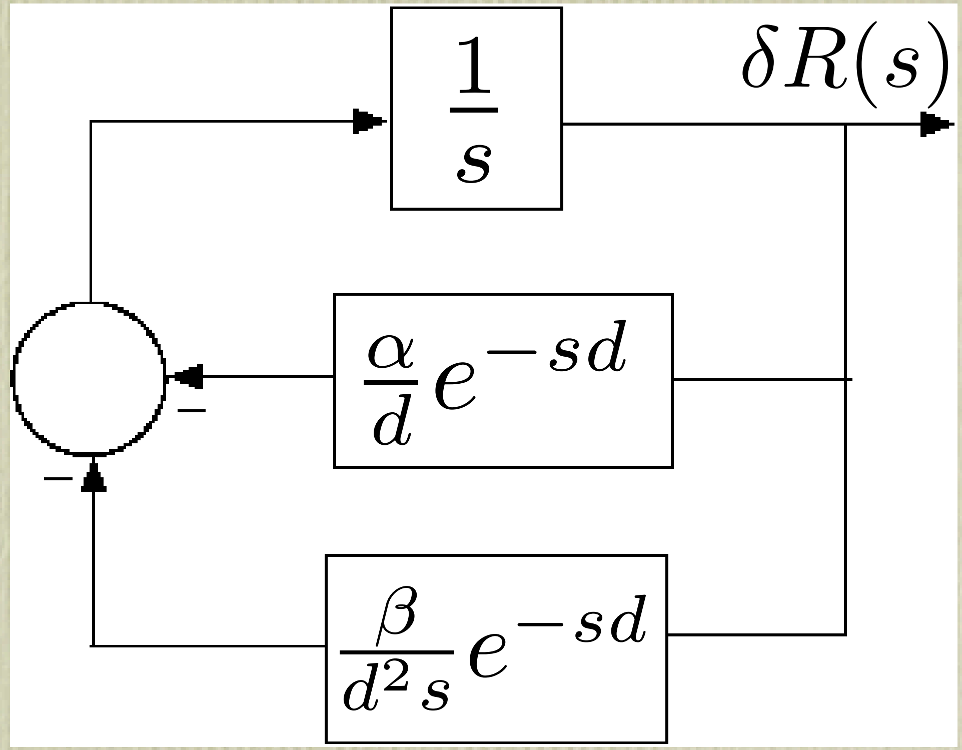
$$(R^*, q^*) = (\tfrac{C}{N}, 0)$$

Linearization

$$\delta\dot{q}(t) = N \times \delta R(t - d)$$

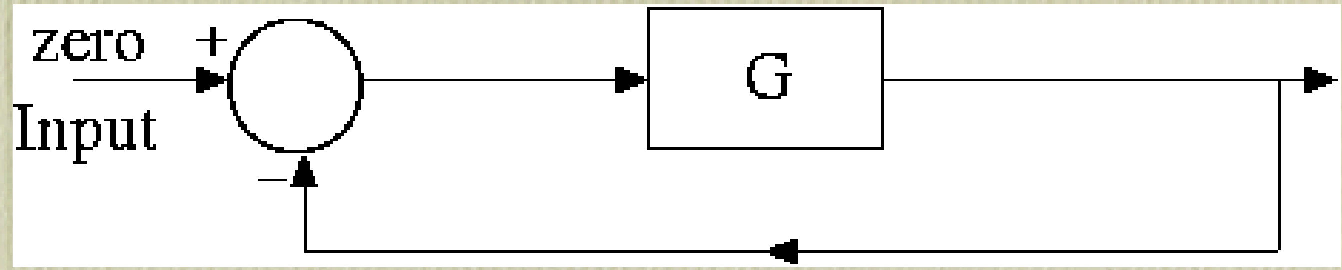
$$\delta\dot{R}(t) = -\tfrac{\alpha}{d}\delta R(t - d) - \tfrac{\beta}{Nd^2}\delta q(t)$$

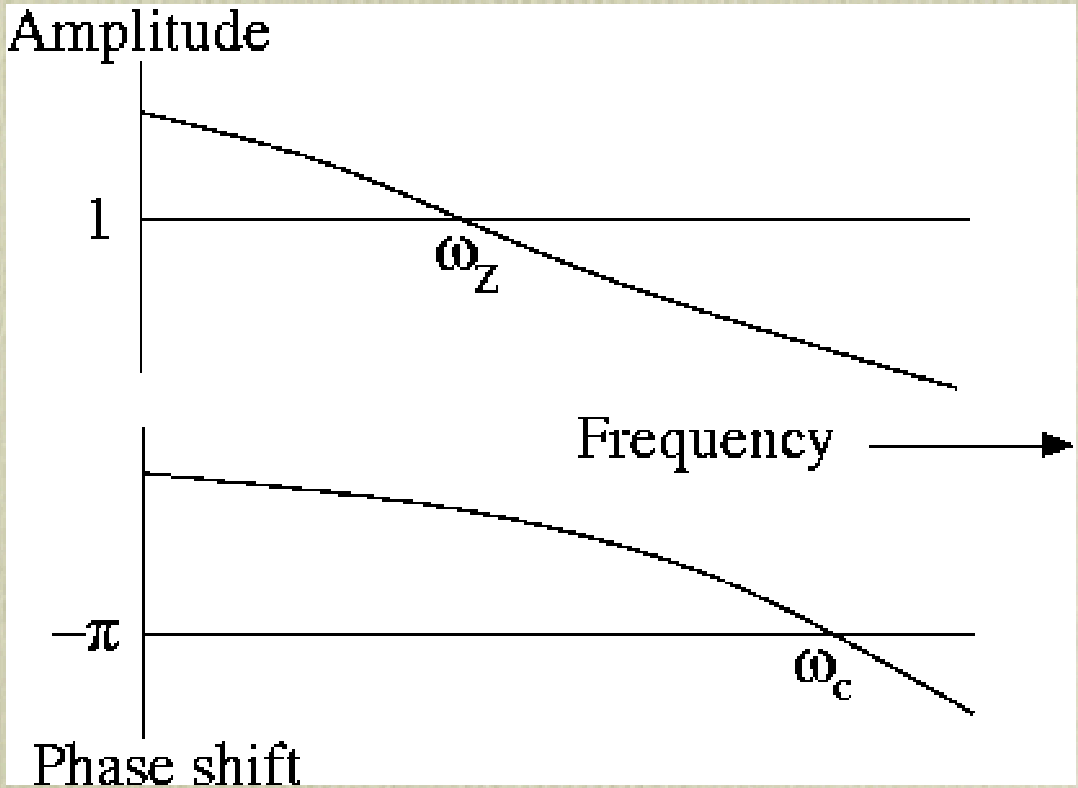
Open Loop Transfer Function



$$G(s) = \frac{e^{-sd}(\alpha s d + \beta)}{s^2 d^2}$$

Bode Plots: Review



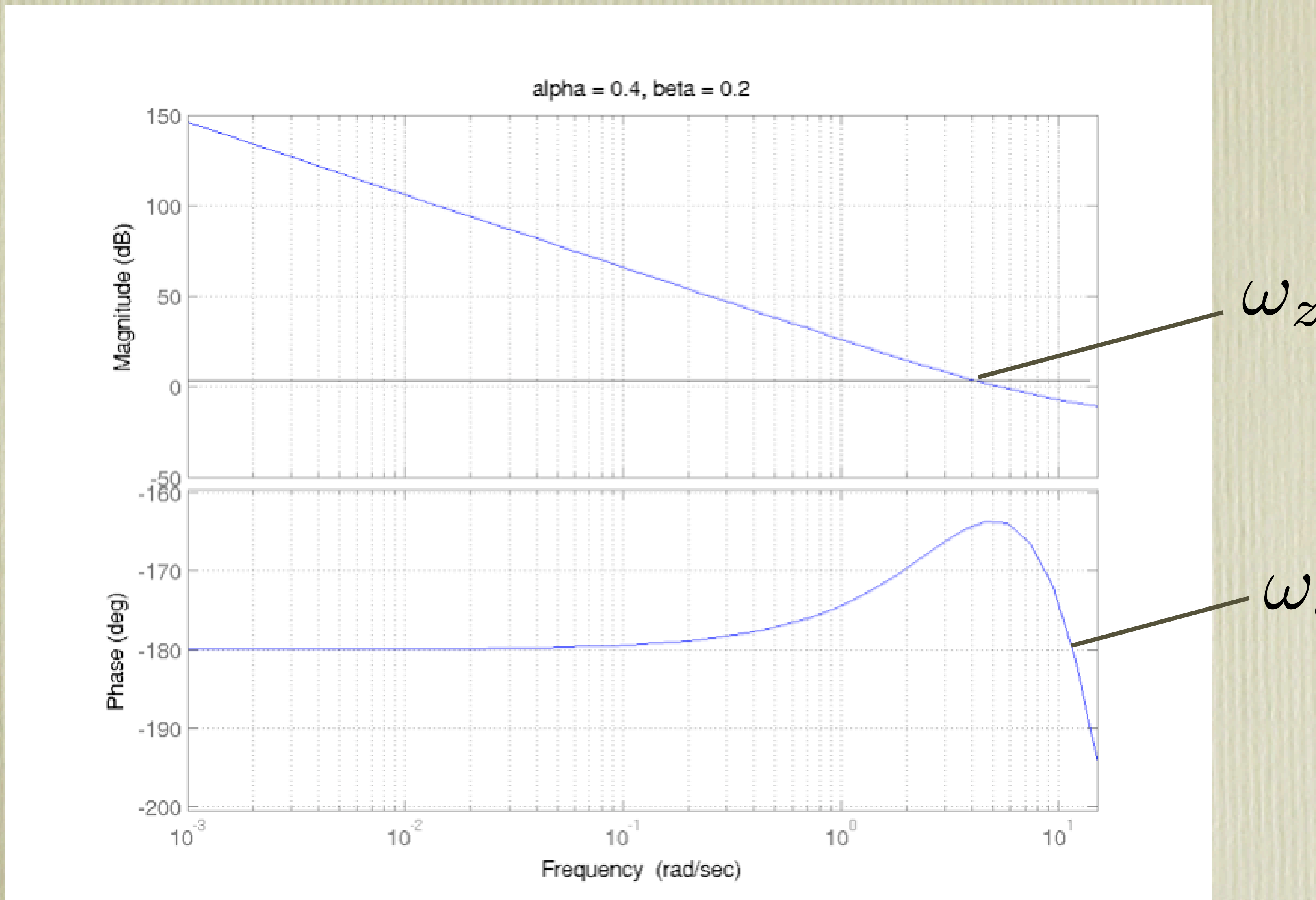


Stability: $\omega_z < \omega_c$

Bode Plot: RCP

$$|G(j\omega)| = |\frac{e^{-j\omega d}(\beta + j\alpha d\omega)}{(j\omega)^2 d^2}|$$

$$\angle G(j\omega) = -\omega d + \tan^{-1}(\frac{\omega \alpha d}{\beta}) - \pi$$



Stability Region (α, β)

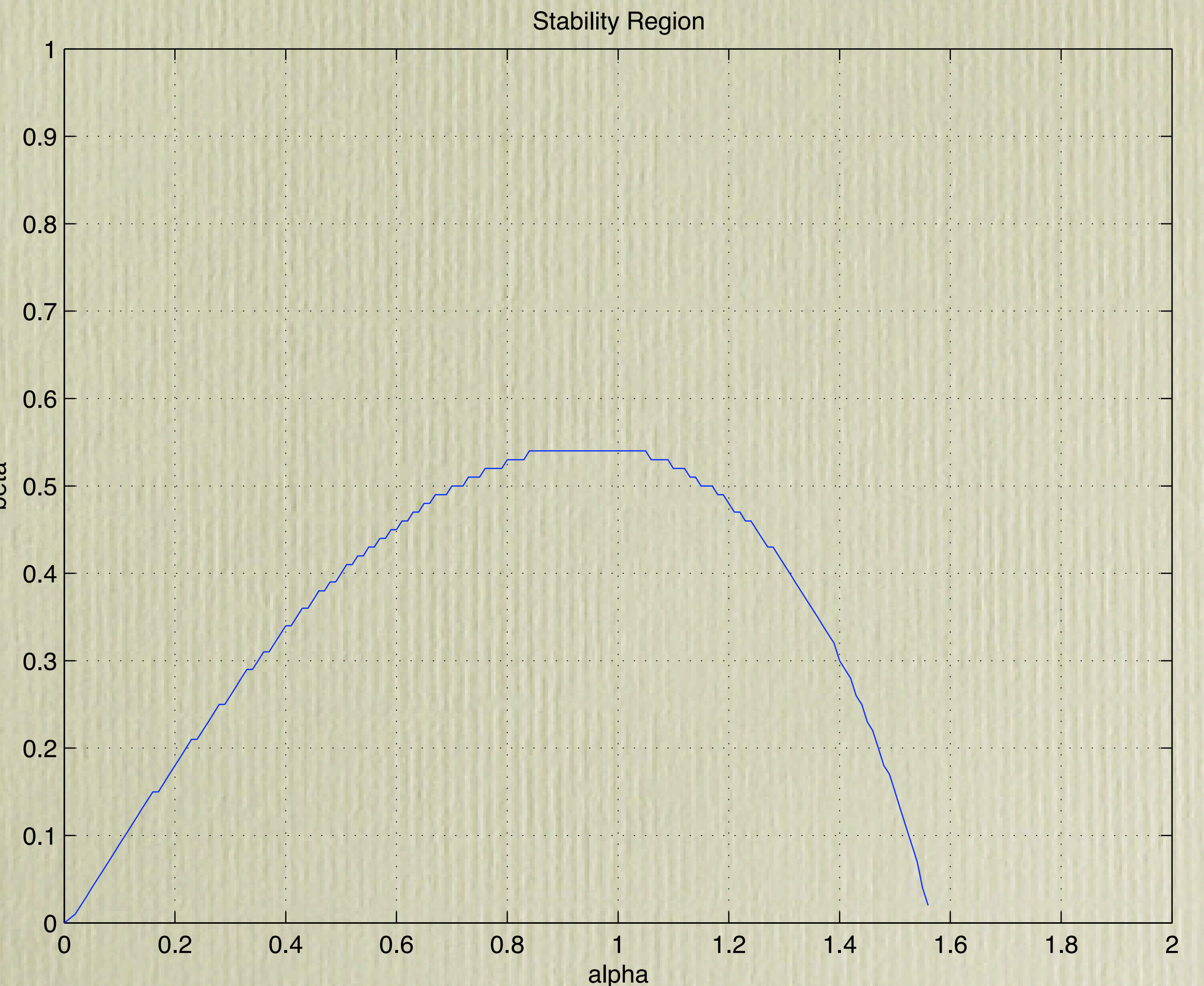
$$|G(j\omega)| = 1 \Longrightarrow \omega_z = \frac{1}{d}\sqrt{\frac{\alpha^2 + \sqrt{\alpha^4 + 4\beta^2}}{2}}$$

$$\angle G(j\omega) = -\pi \Longrightarrow \omega_c = \frac{\beta}{\alpha d}\tan(\omega_c d); \frac{\alpha}{\beta} > 1$$

For stability: $\omega_z < \omega_c$

Stability Region (α, β)

Independent of N, C, d

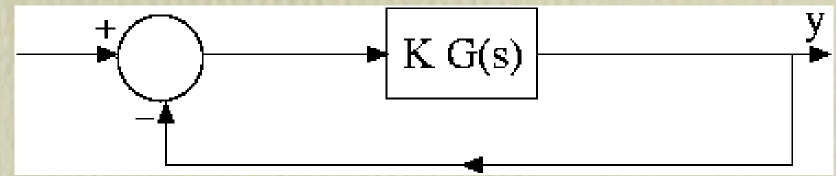


β

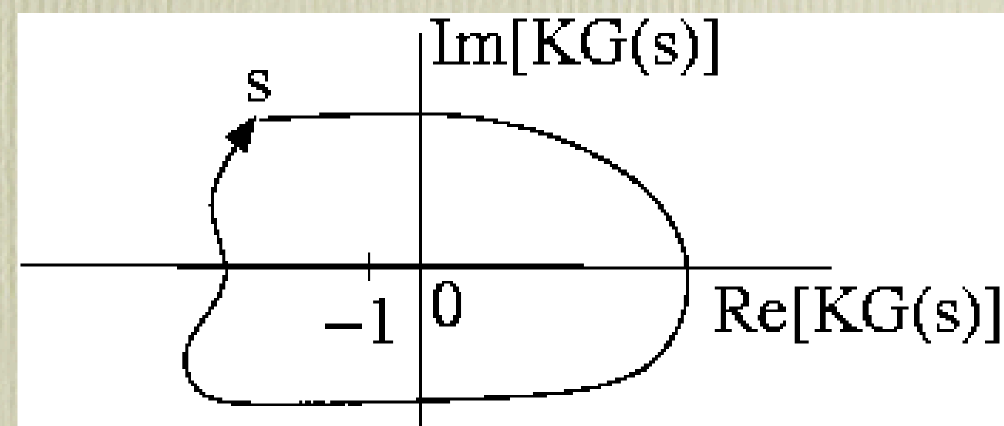
α

Nyquist Stability Criterion: Review

- Plot $KG(s); -j\infty \leq s \leq +j\infty$



- N = # clockwise/anti-clockwise encirclements of -1
- P = # unstable poles of G(s)
- Z = # unstable closed loop poles = N+P
- For stability: Z = 0



Nyquist Stability Criterion: RCP

$$KG(s) = \alpha \; \frac{e^{-sd} sd}{s^2 d^2 + \beta e^{-sd}}$$

Pade approximation for e^{-sd}

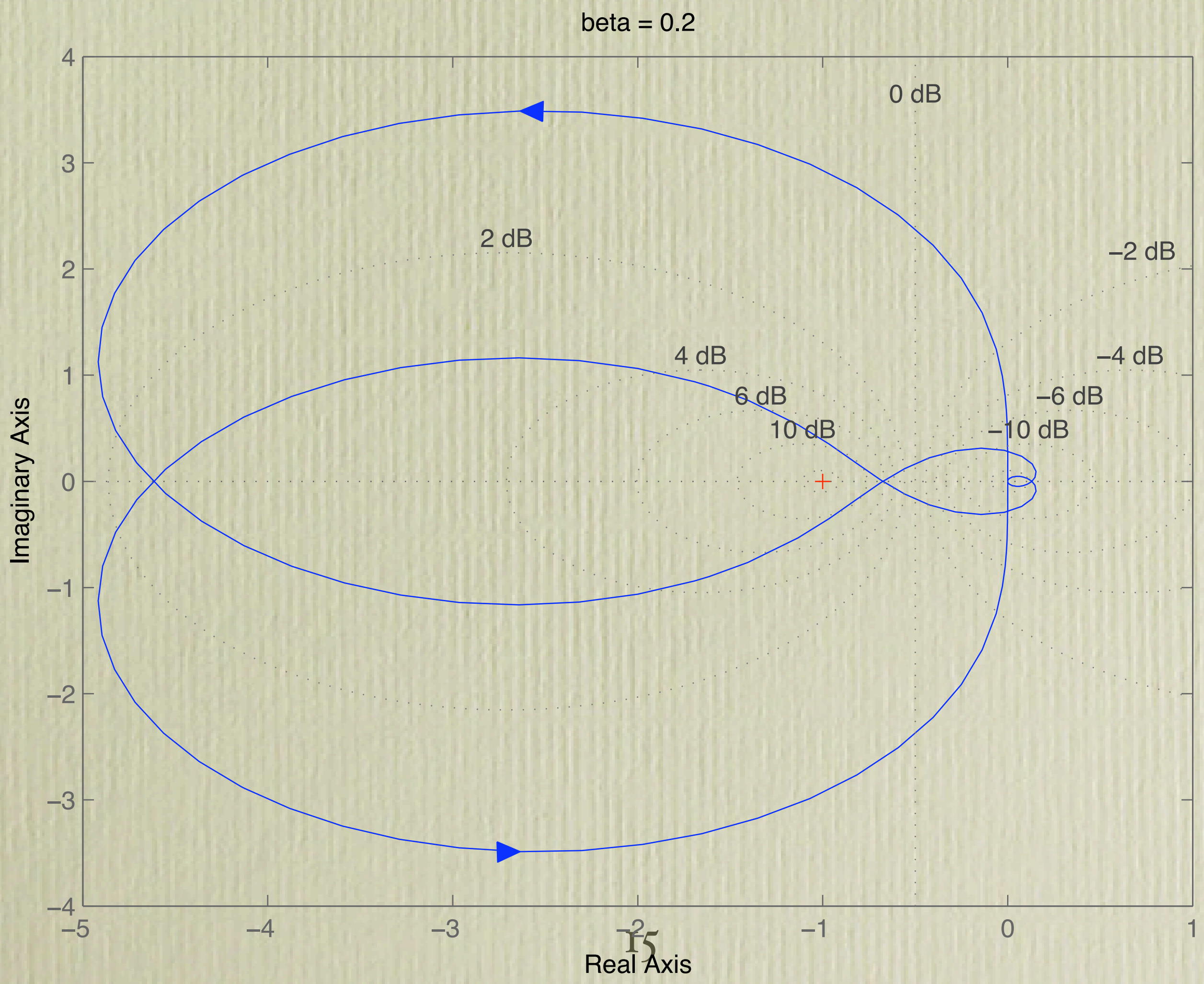
$$KG(s) = \alpha \; \frac{\frac{d^3}{12} s^3 - \frac{d^2}{2} s^2 + ds}{\frac{d^4}{12} s^4 + \frac{d^3}{2} s^3 + (d^2 + \beta \frac{d^2}{12}) s^2 - \frac{d\beta}{2} s + \beta}$$

Nyquist Stability Criterion: RCP

$$\beta = 0.2,\ \ P = 2,\ \ N = -2$$

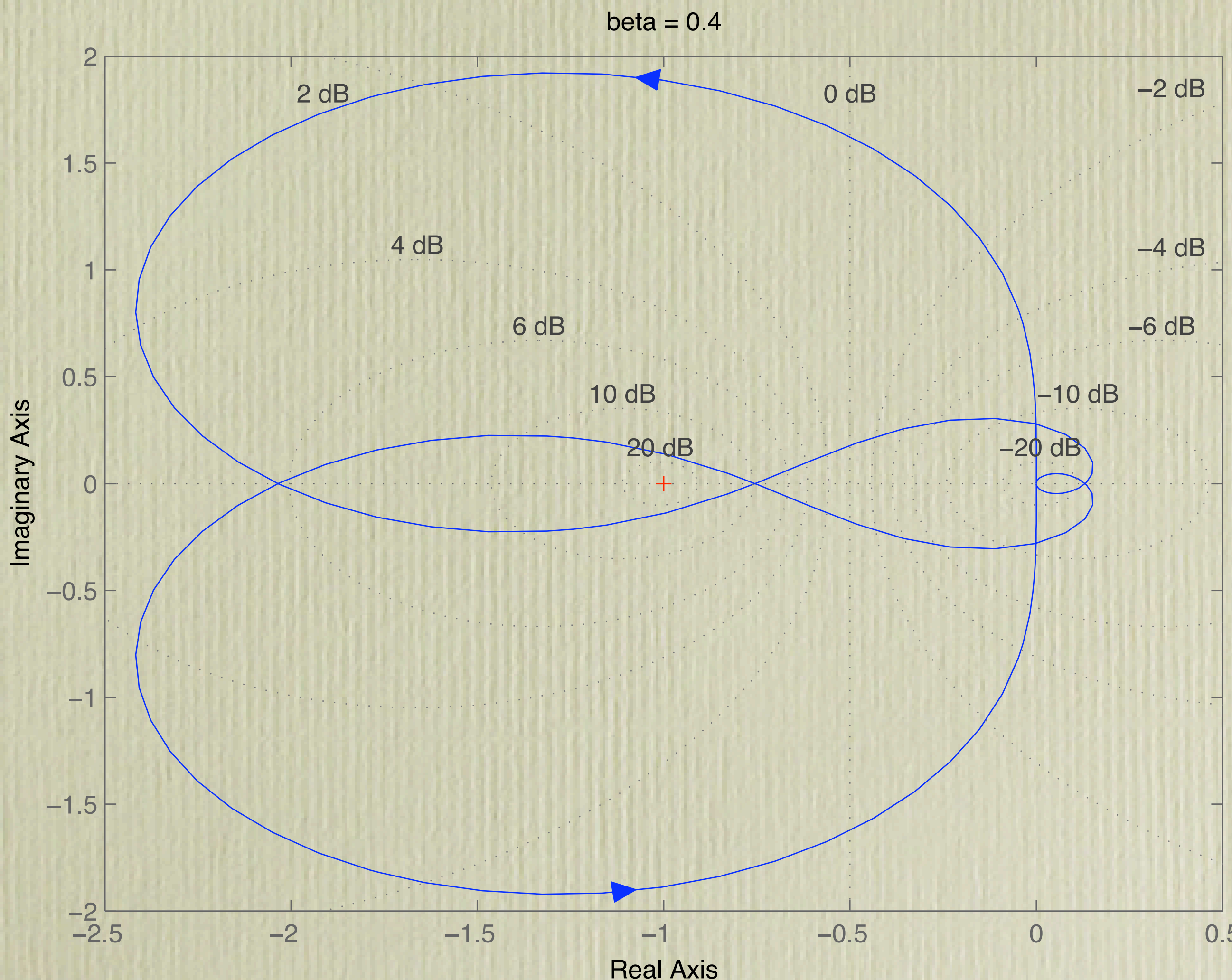
$$Z = N + P = 0 \text{ if } -2.91 < -\frac{1}{\alpha} < -0.696$$

$$0.3436 < \alpha < 1.436$$



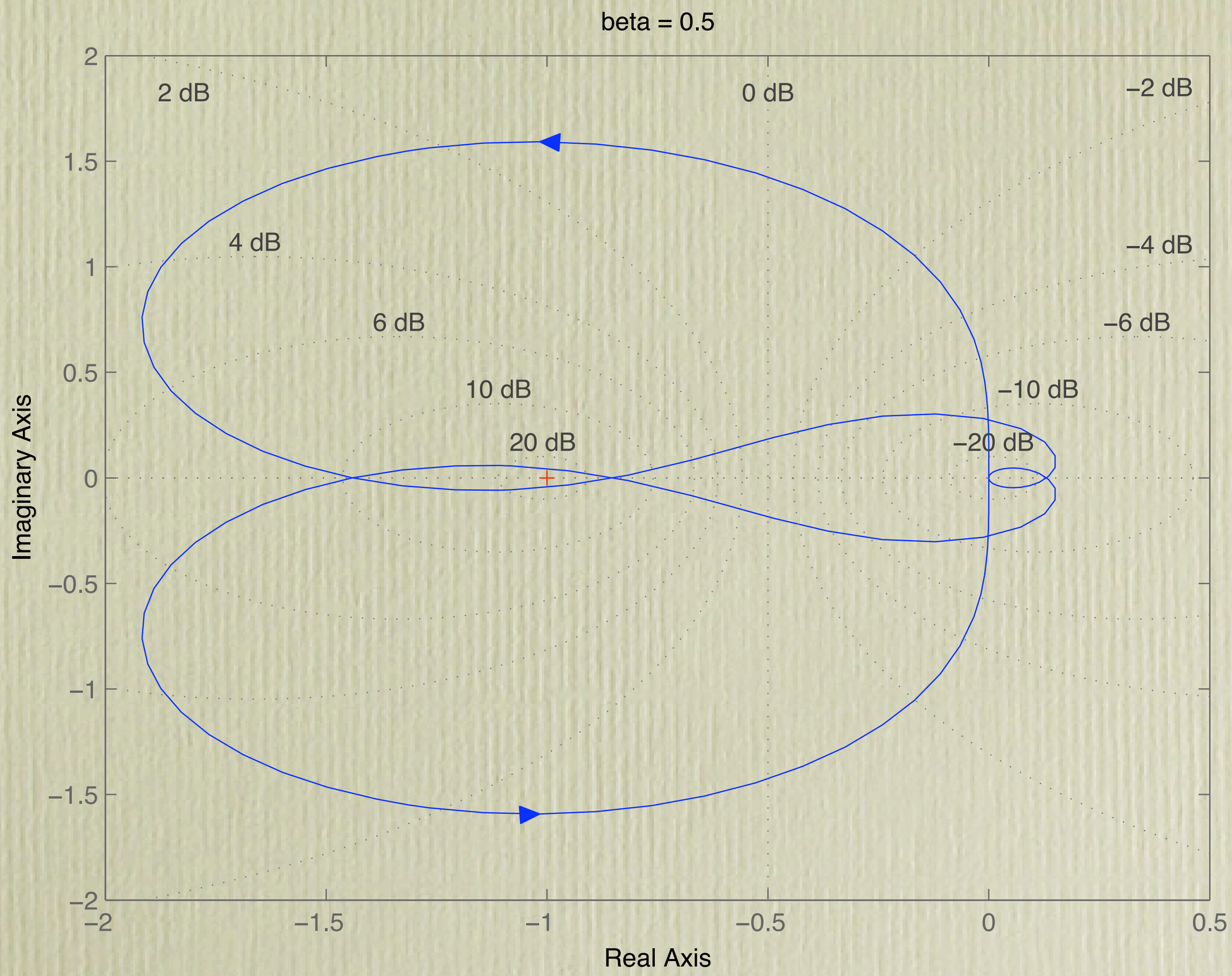
Nyquist Stability Criterion: RCP

$$\beta = 0.4;\ \ 0.492 < \alpha < 1.32$$



Nyquist Stability Criterion: RCP

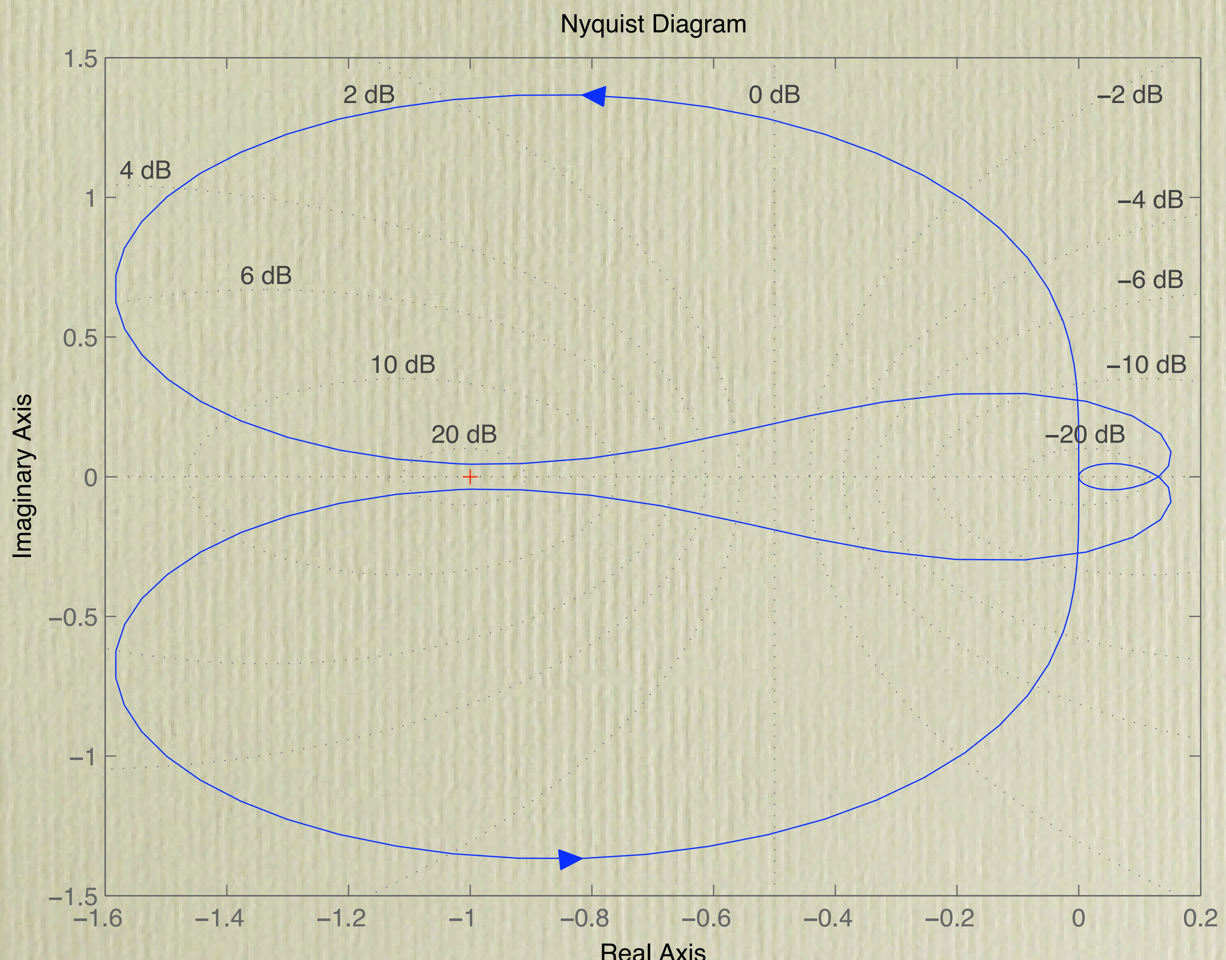
$$\beta = 0.5;\ 0.709 < \alpha < 1.149$$



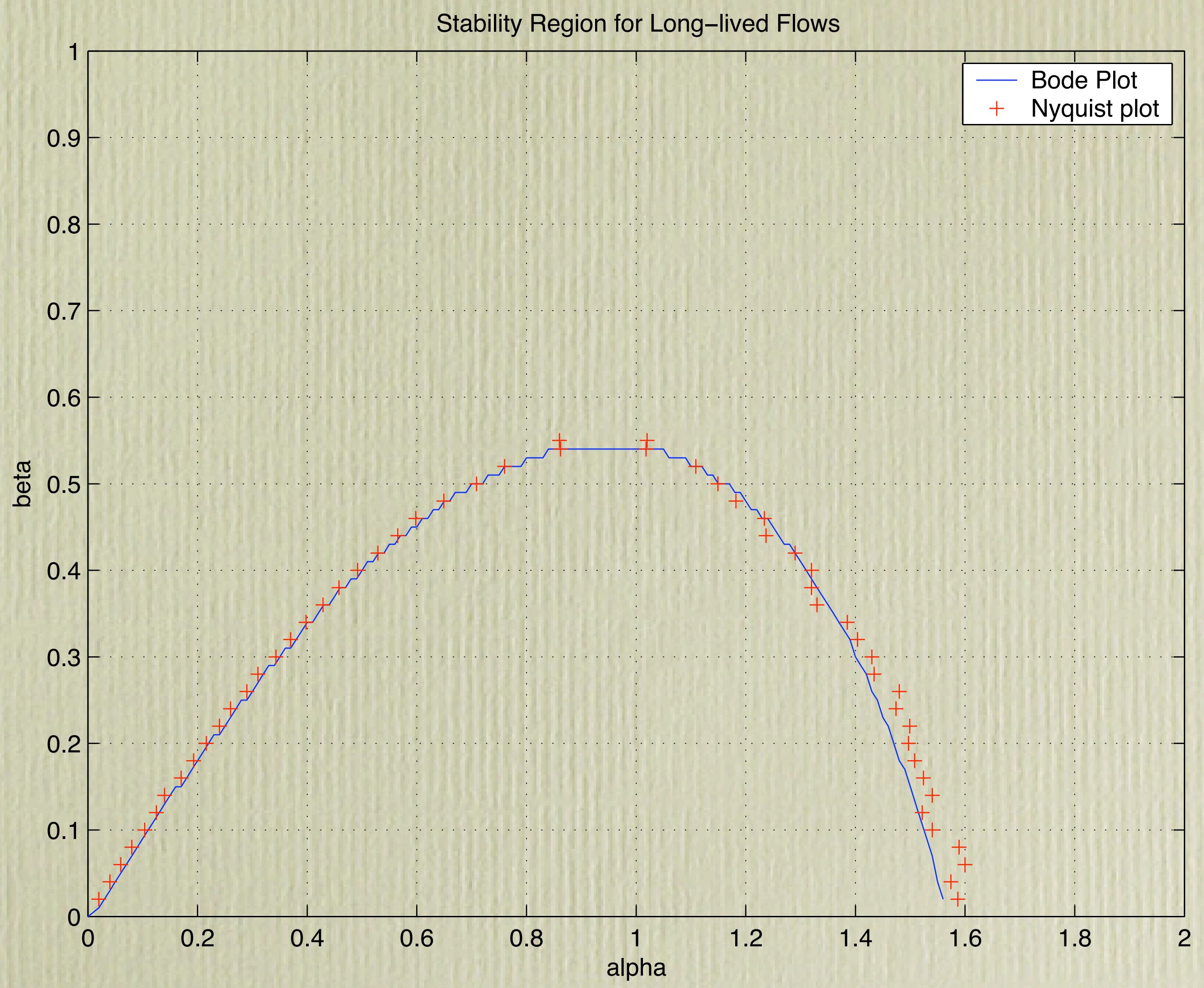
Nyquist Stability Criterion: RCP

$$\beta = 0.6,\ \ P = 2,\ \ Z = N + P \geq 1$$

No stability region

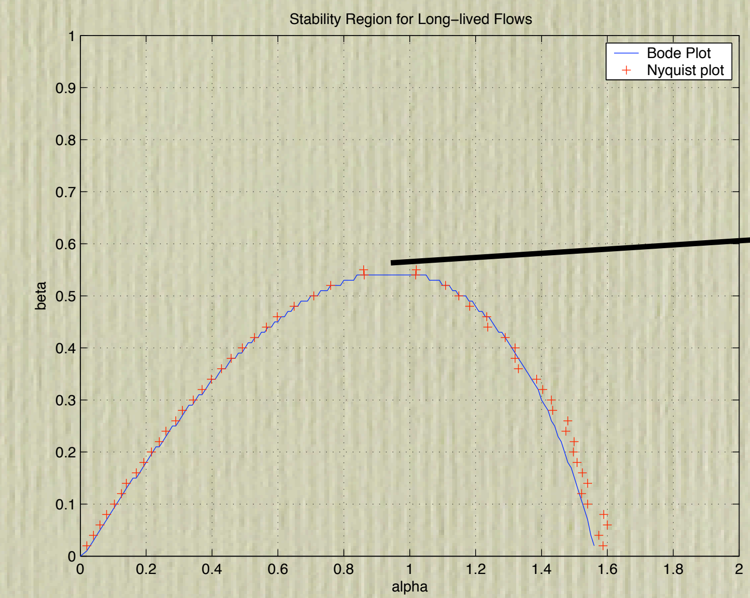


Stability Region



Stability Region

$$R(t) = R(t-T)[1 + \frac{\frac{T}{d}[\alpha(C-y(t))-\beta\frac{q(t)}{d}]}{C}]$$

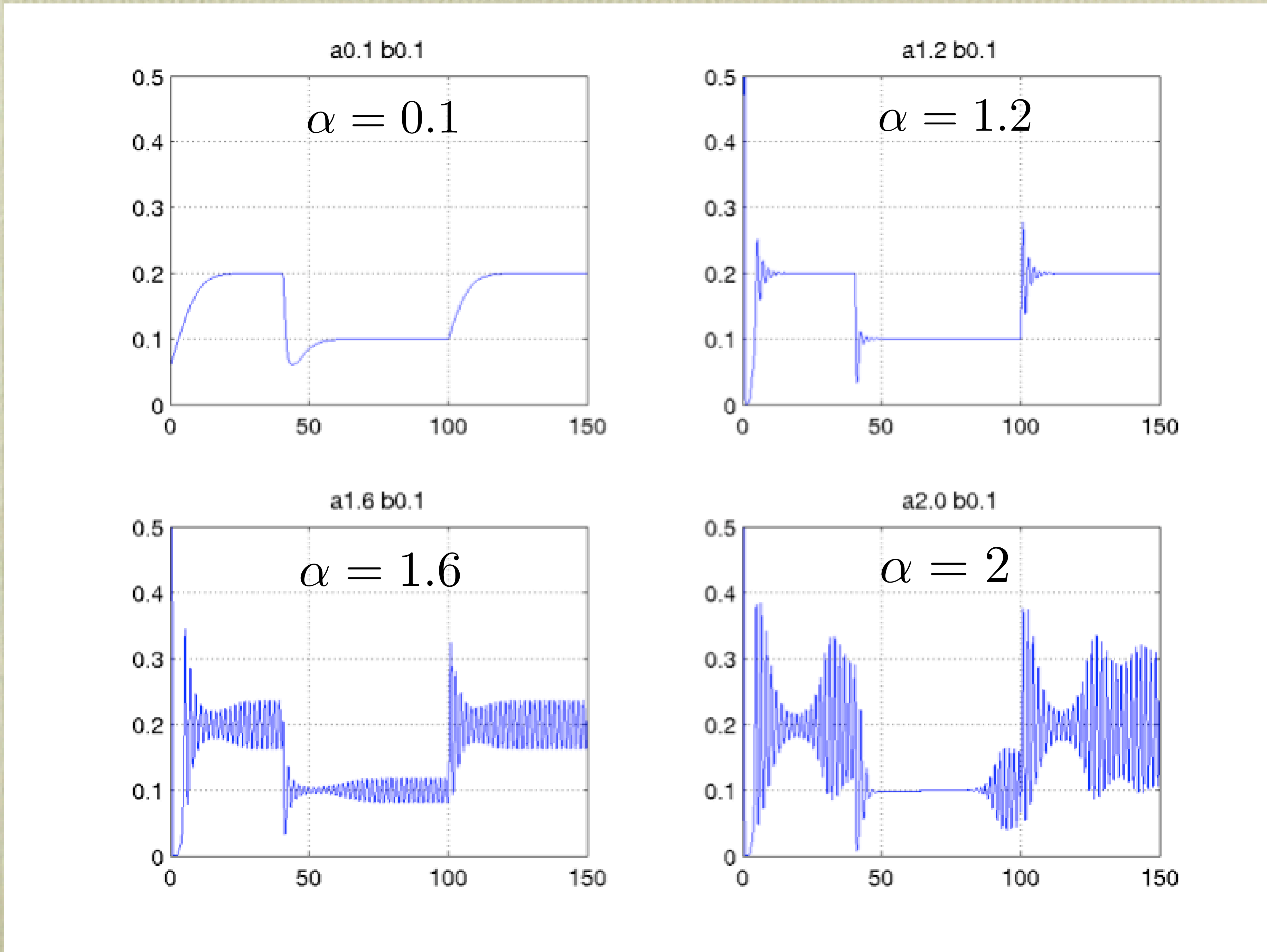


Why the cap?

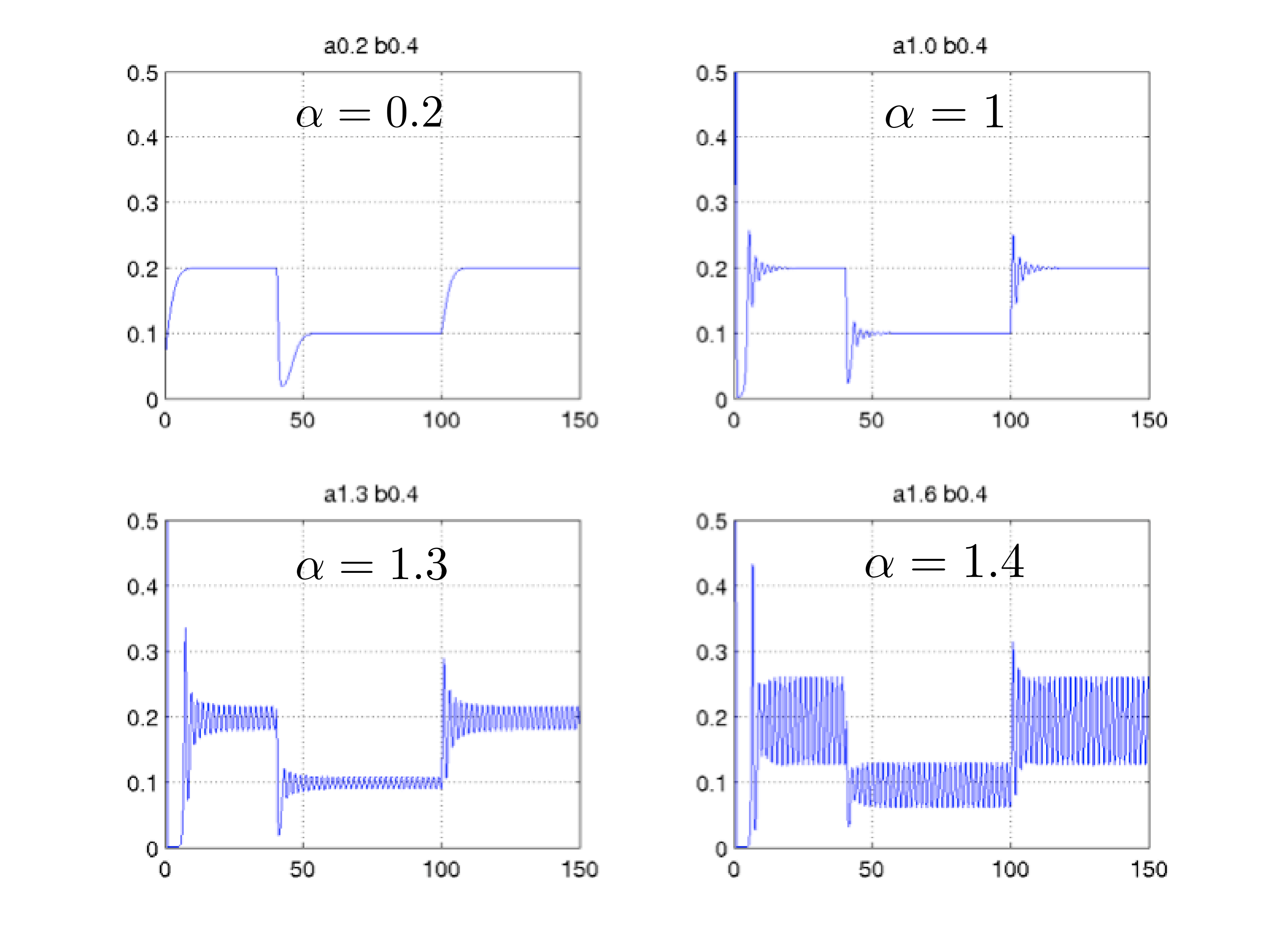
We ignored the non-linearity

$$\dot{q}(t) = \begin{cases} [y(t) - C] \text{ if } q(t) > 0 \\ [y(t) - C]^{+} \text{ if } q(t) = 0 \end{cases}$$

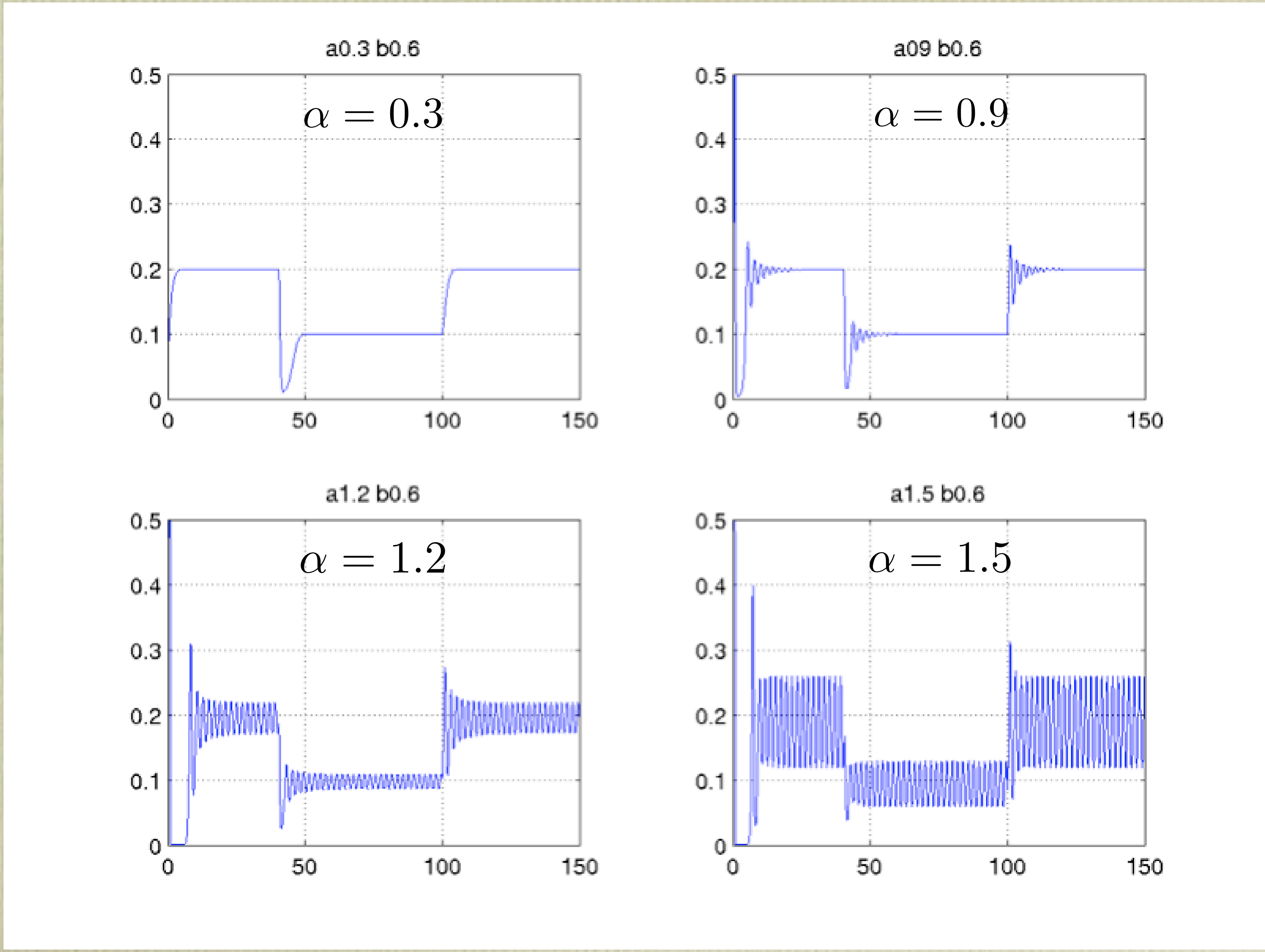
Long Flow Simulations $\beta = 0.1$



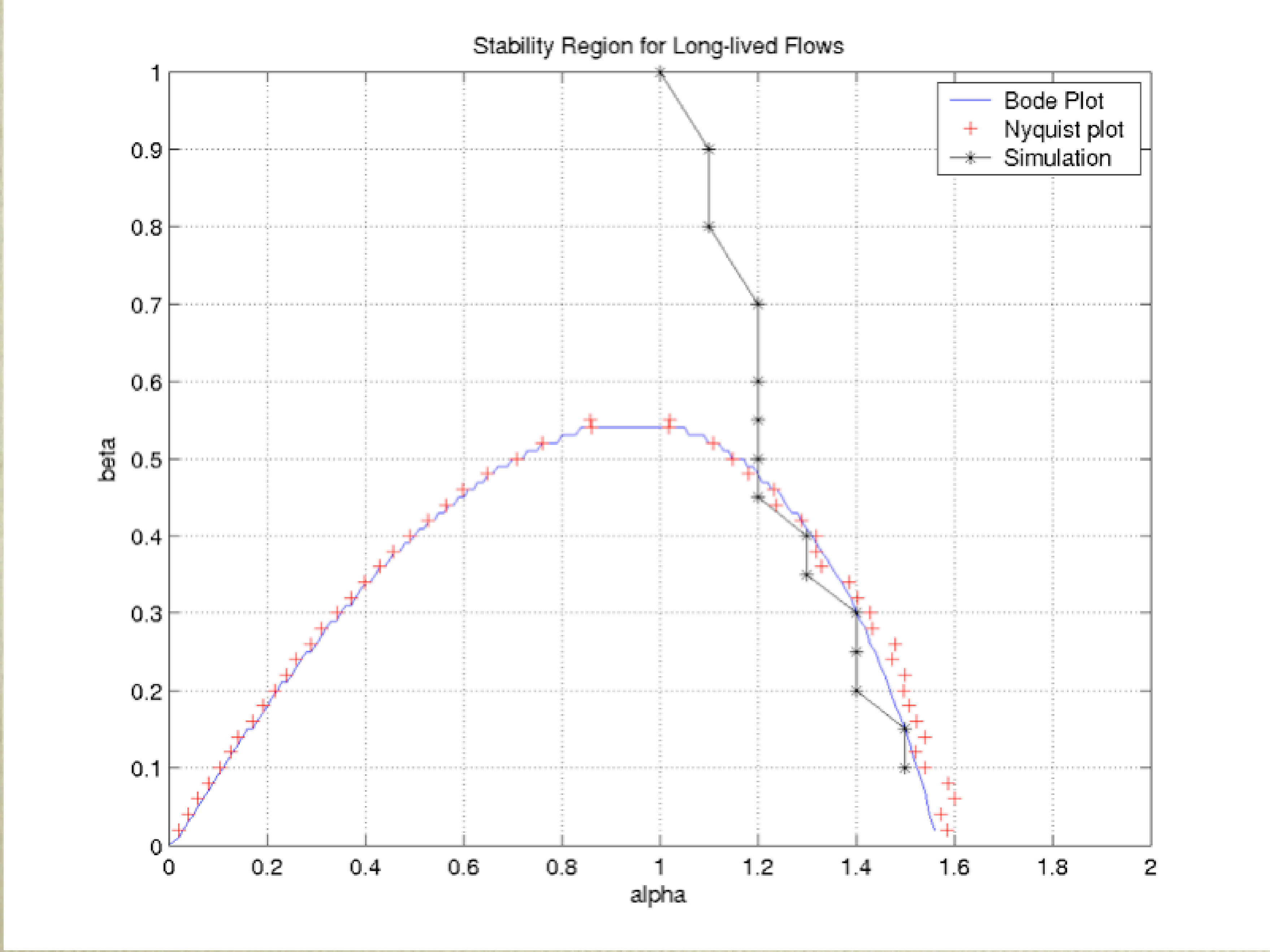
Long Flow Simulations $\beta = 0.4$



Long Flow Simulations $\beta = 0.6$



Stability Region



Ongoing Work

- Relation between stability of linearized system to that of the non-linear system ?
- Stability of the non-linear system?
- Stability of non-linear system independent of N, C, d?
- Describing functions?