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APPENDIX 1

Arbiter Synchronization for Single-Iteration SLIP

In this appendix, we find an approximate expression for the expected number of synchronized output schedulers, $E[S(t)]$.

We partition the set of switch inputs $\mathbf{I} = \{1, \ldots, N\}$ into two subsets at time t: $\mathbf{A}(t)$, the set of inputs that are matched and $\mathbf{B}(t)$, the set of inputs that are not matched. If the arrival rate averaged across all inputs is λ, then for a sustainable and stationary ergodic arrival process the expected match size is λN and on average, λN inputs will send a cell. Clearly then, the expected size of $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are

$$E[|\mathbf{A}(t)|] = \lambda N \quad , \qquad E[|\mathbf{B}(t)|] = (1-\lambda) N = \bar{\lambda} N. \tag{1}$$

Similarly, we partition the set of switch inputs $\mathbf{O} = \{1, \ldots, N\}$ into two subsets at time t: $\mathbf{A}^O$(t)and $\mathbf{B}^O$(t). $\mathbf{A}^O$(t) is the set of outputs that are matched to inputs in $\mathbf{A}$(t), and $\mathbf{B}^O$(t) are the outputs not matched at time t.

As a result of the matching at time t, the set $\mathbf{A}(t)$ is transformed into the set $\tilde{\mathbf{A}}(t+1)$, the set of inputs that the outputs in $\mathbf{A}^O$(t) point to at time t+1. Each element in $\mathbf{A}(t)$ is unique, and because

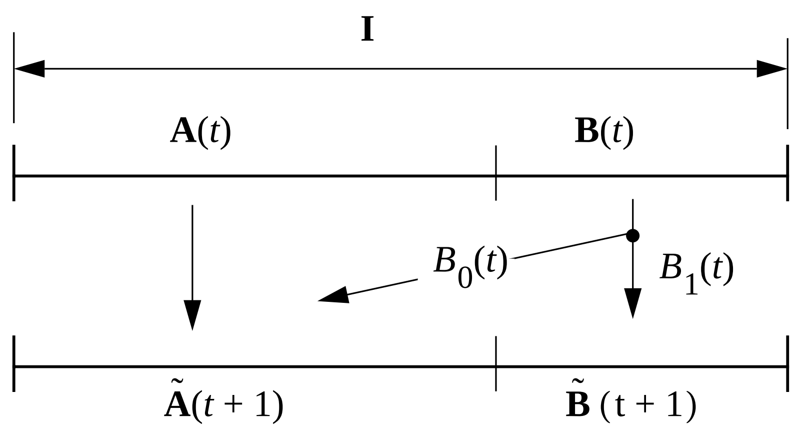


FIGURE A1.1 Mapping of matched and unmatched inputs at time *t*, to modified sets at time *t*+1.

they were matched at time t, each element mapped from $\mathbf{A}(t)$ into $\tilde{\mathbf{A}}(t+1)$ is also unique. The expected size of $\tilde{\mathbf{A}}(t+1)$ is

$$E[|\tilde{\mathbf{A}}(t+1)|] = E[|\tilde{\mathbf{A}}(t)|] = E[|\mathbf{A}(t)|] = \lambda N. \tag{2}$$

Because none of its elements are matched, the set $\mathbf{B}(t)$ is unchanged, i.e. $\tilde{\mathbf{B}}(t+1) = \mathbf{B}(t)$.

To determine $E[S(t+1)]$, the expected number of synchronized output arbiters at time t+1, we must find the number of elements in $\tilde{\mathbf{A}}(t+1)$ that are still unique and the number that clash with elements mapped from $\mathbf{B}(t)$. Without loss of generality, and to simplify our calculations, we assume that a one-to-one mapping is applied to $\tilde{\mathbf{A}}(t+1)$ such that $\tilde{\mathbf{A}}(t+1) = \mathbf{A}(t)$ and hence $\tilde{\mathbf{B}}(t+1) \neq \mathbf{B}(t)$. As before, the elements of $\tilde{\mathbf{A}}(t+1)$ are unique, and we can think of the elements of $\tilde{\mathbf{B}}(t+1)$ as randomly distributed over **I**. This is shown in Figure A1.1.

To find $E[S(t+1)]$, we partition $\mathbf{B}(t)$ into $B_0(t)$ elements that are mapped into $\tilde{\mathbf{A}}(t+1)$, and $B_1(t)$ elements that are mapped into $\tilde{\mathbf{B}}(t+1)$.

Finally, we define $U_A(t+1)$ and $U_B(t+1)$ as the number of unique elements in $\tilde{\mathbf{A}}(t+1)$ and $\tilde{\mathbf{B}}(t+1)$ respectively, and

$$E[S(t+1)] = N - E[U_A(t+1)] - E[U_B(t+1)] \quad . \tag{3}$$

If we assume that under the mapping the elements of $\mathbf{B}(t)$ are *uniformly* distributed in $\mathbf{I}$, then

$$\begin{aligned} &E\left[U_A(t+1) \middle| \; |\tilde{\mathbf{A}}(t+1)|, B_0(t)\right] = |\tilde{\mathbf{A}}(t+1)| \cdot \left(\frac{|\tilde{\mathbf{A}}(t+1)| - 1}{|\tilde{\mathbf{A}}(t+1)|}\right)^{B_0(t)} \\ &\therefore \; E[U_A(t+1) \,|\, |\mathbf{B}(t)|, B_1(t)] = (N - |\mathbf{B}(t)|) \cdot \left(\frac{(N - |\mathbf{B}(t)|) - 1}{(N - |\mathbf{B}(t)|)}\right)^{|\mathbf{B}(t)| - B_1(t)} \end{aligned} \tag{4}$$

and,

$$\begin{aligned} &E\left[U_B(t+1) \,|\, |\tilde{\mathbf{B}}(t)|, B_1(t)\right] = B_1(t) \cdot \left(\frac{|\tilde{\mathbf{B}}(t+1)| - 1}{|\tilde{\mathbf{B}}(t+1)|}\right)^{B_1(t) - 1} \\ &\therefore \; E\left[U_B(t+1) \,|\, |\mathbf{B}(t)|, B_1(t)\right] = B_1(t) \cdot \left(\frac{|\mathbf{B}(t)| - 1}{|\mathbf{B}(t)|}\right)^{B_1(t) - 1} \end{aligned} \tag{5}$$

Hence,

$$\begin{aligned} &E\left[S(t+1) \,|\, |\mathbf{B}(t)|, B_1(t)\right] = \\ &N - (N - |\mathbf{B}(t)|) \cdot \left(\frac{(N - |\mathbf{B}(t)|) - 1}{(N - |\mathbf{B}(t)|)}\right)^{|\mathbf{B}(t)| - B_1(t)} - B_1(t) \cdot \left(\frac{|\mathbf{B}(t)| - 1}{|\mathbf{B}(t)|}\right)^{B_1(t) - 1} . \end{aligned} \tag{6}$$

To find $E[S(t+1)]$ we need to know the distributions of $|\mathbf{B}(t)|$ and $B_1(t)$. Unfortunately, both random variables depend on the traffic arrival pattern. Furthermore, we cannot use Jensen's inequality to bound Eq. 6 from below or above. This is because Eq. 4 is a concave function of $|\mathbf{B}(t)|$ and $B_1(t)$ whereas Eq. 5 is convex.

However, simulations with a variety of arrival patterns indicate that $E[S(t+1)]$ is relatively insensitive to traffic statistics. We therefore approximate the random variables with $|\mathbf{B}(t)| \approx E[|\mathbf{B}(t)|] = \bar{\lambda}N$ and $B_1(t) \approx E[B_1(t)] = \bar{\lambda}^2 N$.

This leads us to the approximation,

$$\mathrm{E}\,[S(t)] \approx N - \lambda N\left(\frac{\lambda N - 1}{\lambda N}\right)^{\lambda\bar{\lambda}N} - \bar{\lambda}^2 N\left(\frac{\bar{\lambda}N - 1}{\bar{\lambda}N}\right)^{\bar{\lambda}^2 N - 1} . \qquad (7)$$

APPENDIX 2

Stability of Single-Iteration SLIP Algorithm

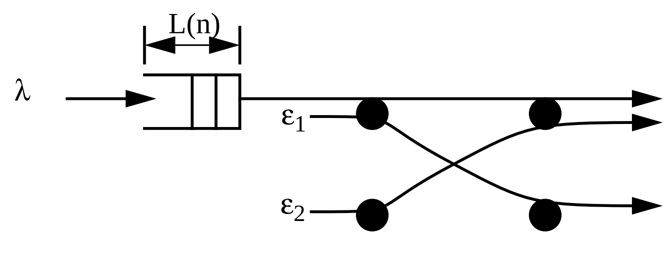


FIGURE A2.1 2x2 switch with a single queue.

1 Single-Step Drift Analysis of 2x2 Switch with 1 Queue

1.1 First Approximation

Consider the switch in Figure A2.1. All three arrival processes are i.i.d. Bernoulli. We wish to find the values of λ, ε_1 and ε_2 for which the switch is *stable*.

Define $\hat{L}$ to be the expected value of $L(n+1)$ (the occupancy of Q(1,1) at time n+1) conditioned on $L(n)$ and $L(n) > 0$

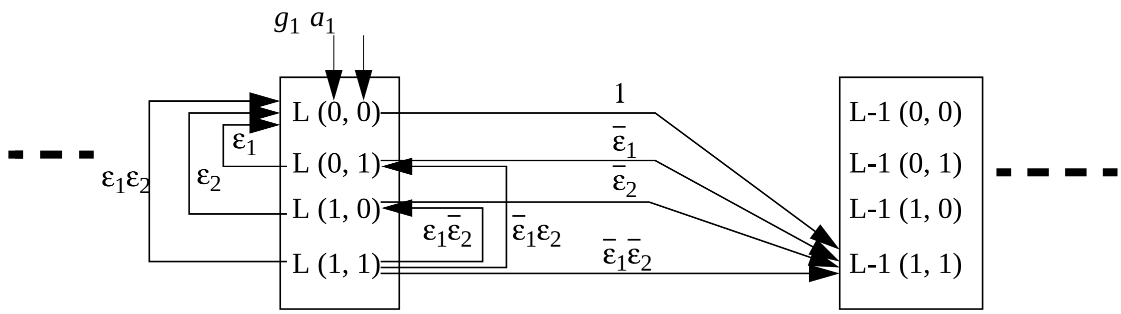
$$\hat{L} = \mathrm{E}\,[L(n+1) |\ L(n), L(n) > 0] \quad . \tag{1}$$

If $\hat{L} - L(n) > 0$ then $L(n)$ has a single-step positive drift which means that $E[L(n)] \rightarrow \infty$ and the switch is unstable.

This system may be described as a discrete-time Markov chain (DTMC) with state

$$\underline{X}_L = (g_1, a_1) \tag{2}$$

where L is the occupancy of $Q(1, 1)$, the value of the pointer g_1 at output 1, and the value of the pointer a_1 at input 1. The evolution of state for the switch using the SLIP algorithm, conditioned on $L(n) > 0$ and $\lambda = 0$ is shown below



where $\bar{\varepsilon}_1 = 1 - \varepsilon_1$, $\bar{\varepsilon}_2 = 1 - \varepsilon_2$. The state transition matrix, P conditioned on $L(n) > 0$ and $\lambda = 0$ is

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \varepsilon_1 & 0 & 0 & \bar{\varepsilon}_1 \\ \varepsilon_2 & 0 & 0 & \bar{\varepsilon}_2 \\ \varepsilon_1\varepsilon_2 & \bar{\varepsilon}_1\varepsilon_2 & \varepsilon_1\bar{\varepsilon}_2 & \bar{\varepsilon}_1\bar{\varepsilon}_2 \end{bmatrix} \tag{3}$$

from which we obtain the steady-state distribution

$$\begin{aligned} \pi(1, 1) &= \frac{1}{1 + \bar{\varepsilon}_1\varepsilon_2 + \varepsilon_1\bar{\varepsilon}_2 + \varepsilon_1\varepsilon_2(1 + \bar{\varepsilon}_1 + \bar{\varepsilon}_2)} \\ \pi(1, 0) &= \varepsilon_1\bar{\varepsilon}_2 \cdot \pi(L, 1, 1) \\ \pi(0, 1) &= \bar{\varepsilon}_1\varepsilon_2 \cdot \pi(L, 1, 1) \\ \pi(0, 0) &= [\varepsilon_1\varepsilon_2(1 + \bar{\varepsilon}_1 + \bar{\varepsilon}_2)] \cdot \pi(L, 1, 1) \end{aligned} \tag{4}$$

From $\underline{\pi}$ we find

$$\hat{L} = L(n) - \left[\frac{1}{1 + \bar{\varepsilon}_1\varepsilon_2 + \varepsilon_1\bar{\varepsilon}_2 + \varepsilon_1\varepsilon_2(1 + \bar{\varepsilon}_1 + \bar{\varepsilon}_2)}\right] \tag{5}$$

which if we consider the arrivals at rate λ gives J, the expected single-step increase function

$$J = \lambda - \left[\frac{1}{1 + 2\varepsilon + \varepsilon^2 - 2\varepsilon^3}\right] \tag{6}$$

where we define $\varepsilon = \varepsilon_1 = \varepsilon_2$ because of the symmetric and identical dependence on ε_1 and ε_2. The unstable region of operation is given by

$$\lambda > \left[\frac{1}{1 + 2\varepsilon + \varepsilon^2 - 2\varepsilon^3}\right]. \tag{7}$$

We can find the maximum positive drift J_{max} (the “most unstable operating point”) by defining

$$\lambda = 1 - \varepsilon - \delta, \qquad \delta < 1 \tag{8}$$

From Eq. 6 we find that $J_{max}\big|_{\delta = 0} \approx 0.098$ which tells us that the drift can be positive for any value of $\delta < 0.098$, or alternatively

$$J > 0 \Rightarrow \lambda + \varepsilon > 1 - 0.098. \tag{9}$$

1.2 Second Approximation

The first approximation assumes that cells arriving at rates ε_1 and ε_2 are discarded if they are not successfully scheduled. However, if unsuccessful cells are queued rather than discarded, they will affect the service rate of $Q(1, 1)$ over multiple cell times. We model this effect by approximating the *busy* and *idle* cycles of input queues $Q(1, 2)$ and $Q(2, 1)$ with a 2-state Markov process, shown in Figure A2.2.

The behavior of $Q(1, 2)$ and $Q(2, 1)$ may be modelled by an M/G/1 queue with an arrival rate λ_2 and service rate $1 - \frac{1}{2}\lambda_1$. From [43] the expected duration of the busy and idle cycles

$$E[B] = \frac{1}{\left(1 - \frac{1}{2}\lambda_1\right) - \lambda_2} = \frac{p}{1 - p} \tag{10}$$

$$E[I] = \frac{1}{\lambda_2} = \frac{q}{1 - q} \tag{11}$$

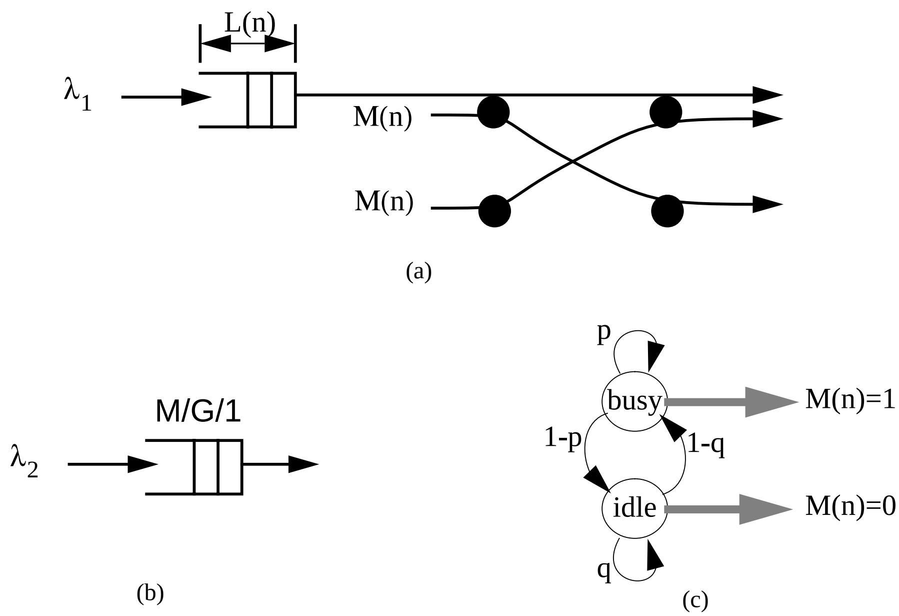


FIGURE A2.2 (a) Approximation of arrivals as an on-off process modulated by a 2-state discrete-time Markov chain, M(n). (b) The arrival process models the busy/idle cycles of input queues Q(1,2) and Q(2,1). (c) The Markov chain alternates between the busy and idle states. In the busy state, the arrival rate is 1. In the idle state the arrival rate is 0.

from which we obtain p and q as functions of λ_1 and λ_2.

To find

$$\hat{L} = \mathrm{E}\,[L(n+1) \mid L(n), L(n) > 0] \tag{12}$$

we model the evolution of the system using a DTMC with state

$$\underline{X} = (g_1, a_1, s_1, s_2) \tag{13}$$

where s_i is the state (*busy* or *idle*) of the 2-state DTMC modulating the arrival process at input i. The state 16x16 transition matrix is

$P =$ (14)

$s_1, s_2 =$		(B, B)				(B, I)				(I, B)				(I, I)			
		00	01	10	11	00	01	10	11	00	01	10	11	00	01	10	11
(B, B)	00				p^2				$\bar{p}p$				$\bar{p}p$				$\bar{p}\bar{p}$
	01	p^2				$\bar{p}p$				$\bar{p}p$				$\bar{p}\bar{p}$			
	10	p^2				$\bar{p}p$				$\bar{p}p$				$\bar{p}\bar{p}$			
	11	p^2				$\bar{p}p$				$\bar{p}p$				$\bar{p}\bar{p}$			
(B, I)	00				$p\bar{q}$				pq				$\bar{p}\bar{q}$				$\bar{p}q$
	01	$p\bar{q}$				pq				$\bar{p}\bar{q}$				$\bar{p}q$			
	10				$p\bar{q}$				pq				$\bar{p}\bar{q}$				$\bar{p}q$
	11			$p\bar{q}$				pq				$\bar{p}\bar{q}$				$\bar{p}q$	
(I, B)	00				$p\bar{q}$				pq				$\bar{p}\bar{q}$				$\bar{p}q$
	01				$p\bar{q}$				pq				$\bar{p}\bar{q}$				$\bar{p}q$
	10	$p\bar{q}$				pq				$\bar{p}\bar{q}$				$\bar{p}q$			
	11			$p\bar{q}$				pq				$\bar{p}\bar{q}$				$\bar{p}q$	
(I, I)	00				$\bar{q}^2$				$\bar{q}q$				$\bar{q}q$				q^2
	01				$\bar{q}^2$				$\bar{q}q$				$\bar{q}q$				q^2
	10				$\bar{q}^2$				$\bar{q}q$				$\bar{q}q$				q^2
	11				$\bar{q}^2$				qq				qq				q^2

from which we find the steady-state distribution

$$\underline{\pi} = (\pi(B, B, 0, 0)\ldots, \pi(I, I, 1, 1)) \quad . \tag{15}$$

For brevity, we show an example of just one element of the distribution

$$\pi(B, B, 0, 0) = \frac{p(q-1)^2(p^2q^2 - pq^2 + p^3q - 3qp^2 + 5pq - 2q - p^3 + p^2 - 3p + 2)}{(q^2p^3 + p^2q^2 - 3q^2p + q^2 + p^4q - p^3q - p^3q - 2qp^2 + 7pq - 3q - p^4 + p^3 + 2p^2 - 3p + 3)(q - 2 + p)^2} \tag{16}$$

We find $\hat{L}$ from $\underline{\pi}$ and the expression

$$\hat{L} = L(n) - [\pi(B, B, 0, 0) + \pi(B, I, 0, 0) + \pi(I, B, 0, 0) + \pi(I, I, 0, 0) + \pi(B, I, 1, 0) + \pi(I, B, 0, 1) + \pi(I, I, 0, 1) + \pi(I, I, 1, 0) + \pi(I, I, 1, 1)] \tag{17}$$

This leads to an expression for the single-step drift function J as the ratio of two 10th degree polynomials. As a result, we have only been able to find the conditions on λ_1 and λ_2 numerically such that J is negative and the switch is stable.

2 Matrix Geometric Solution for 2x2 Switch with 1 Queue

Consider again the switch in Figure A2.1. In this section, we find the steady-state distribution function for the queue occupancy, L and find the values of λ and ε necessary for stability. To do this, we use the matrix-geometric technique of Neuts [36]. We define the state

$$\underline{\mathbf{X}} = [\underline{X}_0, \underline{X}_1, \underline{X}_2, \ldots] \tag{18}$$

where $\underline{X}_L = \{(L, g_1, a_1); g_1 \in \{0, 1\}, a_1 \in \{0, 1\}\}$ and wish to solve the infinite system of linear equations

$$\Pi = \Pi\tilde{\mathbf{P}}, \qquad \Pi\underline{e} = 1 \tag{19}$$

where

$$\Pi = [\underline{\Pi}_0, \underline{\Pi}_1, \underline{\Pi}_2, \ldots], \qquad \underline{\Pi}_i = \text{steady-state distribution of } \underline{X}_i, \tag{20}$$

$\underline{e}$ is the column vector with all its elements equal to 1, and the transition probability matrix is of the form

$$\tilde{\mathbf{P}} = \begin{bmatrix} B_0 & A_0 & 0 & \ldots \\ B_1 & A_1 & A_0 & \ldots \\ B_2 & A_2 & A_1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}. \tag{21}$$

In this example, $\tilde{\mathbf{P}}$ is most easily understood when separated into two parts, conditioned on whether or not an arrival occurs

$$\tilde{\mathbf{P}} = \lambda\tilde{\mathbf{P}}_\lambda + \bar{\lambda}\tilde{\mathbf{P}}_{\bar{\lambda}} \tag{22}$$

where $\tilde{\mathbf{P}}_\lambda$ and $\tilde{\mathbf{P}}_{\bar{\lambda}}$ are

$$\tilde{\mathbf{P}}_\lambda = \left[\begin{array}{c|cccc|cccc|cccc|c}
 & 00 & 01 & 10 & 11 & 00 & 01 & 10 & 11 & 00 & 01 & 10 & 11 & \\ \hline
00 & & & & 1 & & & & & & & & & \\
01 & & & & \bar\varepsilon & \varepsilon & & & & & & & & \\
10 & & & & \bar\varepsilon & \varepsilon & & & & & & & & \\
11 & & & & \bar\varepsilon^2 & \varepsilon^2 & \varepsilon\bar\varepsilon & \varepsilon\bar\varepsilon & & & & & & \\ \hline
00 & & & & & & & & 1 & & & & & \\
01 & & & & & & & & \bar\varepsilon & \varepsilon & & & & \\
10 & & & & & & & & \bar\varepsilon & \varepsilon & & & & \\
11 & & & & & & & & \bar\varepsilon^2 & \varepsilon^2 & \varepsilon\bar\varepsilon & \varepsilon\bar\varepsilon & & \\ \hline
00 & & & & & & & & & & & & 1 & \\
01 & & & & & & & & & & & & \bar\varepsilon & \varepsilon \\
10 & & & & & & & & & & & & \bar\varepsilon & \varepsilon \\
11 & & & & & & & & & & & & \bar\varepsilon^2 & \varepsilon^2 \ \ \varepsilon\bar\varepsilon \ \ \varepsilon\bar\varepsilon \\ \hline
 & & & & & & & & & & & & & \ddots
\end{array}\right]$$

L=0 L=1 L=2

$$\tilde{\mathbf{P}}_{\bar{\lambda}} = \left[\begin{array}{c|cccc|cccc|cccc|c}
 & 00 & 01 & 10 & 11 & 00 & 01 & 10 & 11 & 00 & 01 & 10 & 11 & \\ \hline
00 & 1 & & & & & & & & & & & & \\
01 & \varepsilon & \bar\varepsilon & & & & & & & & & & & \\
10 & \varepsilon & \varepsilon & & & & & & & & & & & \\
11 & \varepsilon^2 & \varepsilon\bar\varepsilon & \varepsilon\bar\varepsilon & \bar\varepsilon^2 & & & & & & & & & \\ \hline
00 & & & & 1 & & & & & & & & & \\
01 & & & & \bar\varepsilon & \varepsilon & & & & & & & & \\
10 & & & & \bar\varepsilon & \varepsilon & & & & & & & & \\
11 & & & & \bar\varepsilon^2 & \varepsilon^2 & \varepsilon\bar\varepsilon & \varepsilon\bar\varepsilon & & & & & & \\ \hline
00 & & & & & & & & 1 & & & & & \\
01 & & & & & & & & \bar\varepsilon & \varepsilon & & & & \\
10 & & & & & & & & \bar\varepsilon & \varepsilon & & & & \\
11 & & & & & & & & \bar\varepsilon^2 & \varepsilon^2 & \varepsilon\bar\varepsilon & \varepsilon\bar\varepsilon & & \\ \hline
 & & & & & & & & & & & & & \ddots
\end{array}\right]$$

Clearly, $\tilde{\mathbf{P}}$ is of the form of Eq. 21.

To find the steady state distribution Π, we use Lemma 1.2.3 and Theorem 1.2.1 of [36]. If $\tilde{\mathbf{P}}$ is positive recurrent, then we can find (using a method outlined on page 9 of [36]) a unique matrix R which satisifies the equation

$$R = \sum_{k=0}^{\infty} R^k A_k \qquad \text{, where } A_k \text{ is as shown in Eq. 21.} \tag{23}$$

such that $\underline{\Pi}_{i+1} = \underline{\Pi}_i R$ for $i \geq 0$, the spectral radius of R, sp(R)<1, the matrix

$$B[R] = \sum_{k=0}^{\infty} R^k B_k \tag{24}$$

is stochastic,

$$\underline{\Pi}_0 (I-R)^{-1} \underline{e} = 1 \text{, and} \tag{25}$$

$$\underline{\Pi}_0 = \underline{\Pi}_0 B[R] \quad . \tag{26}$$

Alternatively, for the Markov chain to be positive recurrent (i.e. for the system to be stable) it is necessary that the spectral radius of R be less than 1.

Solving for R and $B[R]$ we obtain

$$R = g(\varepsilon, \lambda) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1-2\varepsilon\lambda+2\varepsilon^2\lambda & (1-\varepsilon)\varepsilon\lambda & (1-\varepsilon)\varepsilon\lambda & \dfrac{\lambda}{1-\lambda} \\ 1-2\varepsilon\lambda+2\varepsilon^2\lambda & (1-\varepsilon)\varepsilon\lambda & (1-\varepsilon)\varepsilon\lambda & \dfrac{\lambda}{1-\lambda} \\ \varepsilon(3-2\varepsilon-2\lambda+2\varepsilon\lambda) & 1-\varepsilon & 1-\varepsilon & \dfrac{\lambda(2+\varepsilon-2\varepsilon^2-2\varepsilon\lambda+2\varepsilon^2\lambda)}{1-\lambda} \end{bmatrix} \tag{27}$$

$$B[R] = \begin{bmatrix} 1-\lambda & 0 & 0 & \lambda \\ \varepsilon(1-\lambda) & 1-\varepsilon-\lambda+\varepsilon\lambda & 0 & \lambda \\ \varepsilon(1-\lambda) & 0 & 1-\varepsilon-\lambda+\varepsilon\lambda & \lambda \\ \varepsilon^2(1-\lambda) & (1-\varepsilon)\varepsilon(1-\lambda) & (1-\varepsilon)\varepsilon(1-\lambda) & 1-2\varepsilon+\varepsilon^2+2\varepsilon\lambda-\varepsilon^2\lambda \end{bmatrix}, \tag{28}$$

where

$$g(\varepsilon, \lambda) = \frac{\varepsilon\lambda}{(1-2\varepsilon\lambda-\varepsilon^2\lambda+2\varepsilon^3\lambda+2\varepsilon^2\lambda^2-2\varepsilon^3\lambda^2)} . \tag{29}$$

The spectral radius

$$\mathrm{sp}(R) < 1 \Leftrightarrow \lambda < \left[\frac{1}{1+2\varepsilon+\varepsilon^2-2\varepsilon^3}\right] \tag{30}$$

which is identical to the stability requirement of Eq. 7.

We can also solve Eq. 25 and Eq. 26 above to find $\underline{\Pi}_0$. The resulting expression is a vector of elements, each of which is the ratio of two 6th order polynomials in ε and λ from which we can successively generate $\underline{\Pi}_1, \underline{\Pi}_2, \underline{\Pi}_3, \ldots$. We do not repeat these (long) expressions here.

APPENDIX 3

Stability of 2x2 Switch

1 Stability of 2x2 Switch with 3 Active Flows (Theorem 4.2)

In this section, we find sufficient conditions on a scheduling algorithm for a 2x2 switch with 3 active flows such that the switch is stable under all admissible, arrival processes with i.i.d. interarrival times. The switch is illustrated in Chapter 4 Figure 4.6.

1.1 Definitions

Define the vector of queue occupancies

$$\underline{L}(n) = (L_{1,1}(n), L_{1,2}(n), L_{2,1}(n)) \ . \quad (1)$$

We now consider the single-step change in $\underline{L}(n)$ conditioned on whether the switch is in configuration A or B, as shown in Chapter 4 Figure 4.6:

$$\left.\begin{aligned} L_{1,1}(n+1) &= [L_{1,1}(n)-1]^{+}+\eta_1 \\ L_{1,2}(n+1) &= L_{1,2}(n)+\eta_2 \\ L_{2,1}(n+1) &= L_{2,1}(n)+\eta_3 \end{aligned}\right\} \text{Configuration } A$$

$$\left.\begin{aligned} L_{1,1}(n+1) &= L_{1,1}(n)+\eta_1 \\ L_{1,2}(n+1) &= [L_{1,2}(n)-1]^{+}+\eta_2 \\ L_{2,1}(n+1) &= [L_{2,1}(n)-1]^{+}+\eta_3 \end{aligned}\right\} \text{Configuration } B \tag{2}$$

where

$$\eta_i = \begin{cases} 1, & \text{if an arrival occurs at queue i, w.p. } \lambda_i \\ 0, & \text{else} \end{cases} \tag{3}$$

We define the quadratic Lyapunov function

$$V(\underline{L}(n)) = \underline{L}(n)^T Q \underline{L}(n) \geq 0 , \quad \text{where } Q = [q_{ij}], \quad q_{ij} \geq 0 . \tag{4}$$

1.2 Problem statement

If we can find a Q such that $E[V(\underline{L}(n+1)) - V(\underline{L}(n)) \mid \underline{L}(n)] < 0$, then the queue occupancy has a downward drift and the switch is said to be *stable*.

1.3 Solution

As there is no systematic method for finding Q we must guess its form. We assume that Q is symmetric, i.e. $[q_{ij}] = [q_{ji}]$. Further, we guess that if the switch is stable under all admissible offered loads, then it will be marginally stable when $\lambda_2 = \lambda_3 = 1$ and $\lambda_1 = 0$. i.e.

$$E[V(\underline{L}(n+1)) - V(\underline{L}(n)) \mid \underline{L}(n), \lambda_2 = \lambda_3 = 1, \lambda_1 = 0] = 0 . \tag{5}$$

This leads us to the guess

$$Q = a\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \text{ for any integer, } a . \tag{6}$$

This matrix Q leads to a stable switch under the following conditions.

Conditioned on $L_{1,1}(n) > 0, L_{1,2}(n) > 0, L_{2,1}(n) > 0$:

$$\begin{aligned} E[V(\underline{L}(n+1)) - V(\underline{L}(n)) \mid \underline{L}(n), A] &= (-2 + 2\lambda_1 + \lambda_2 + \lambda_3) \\ &\quad (-2 + 2\lambda_1 + \lambda_2 + \lambda_3 + 4L_{1,1}(n) + 2L_{1,2}(n) + 2L_{2,1}(n)) \\ &< 0 \end{aligned} \tag{7}$$

$$\begin{aligned} E[V(\underline{L}(n+1)) - V(\underline{L}(n)) \mid \underline{L}(n), B] &= (-2 + 2\lambda_1 + \lambda_2 + \lambda_3) \\ &\quad (-2 + 2\lambda_1 + \lambda_2 + \lambda_3 + 4L_{1,1}(n) + 2L_{1,2}(n) + 2L_{2,1}(n)) \\ &< 0 \end{aligned} \tag{8}$$

Similarly, conditioned on either $L_{1,2}(n) = 0$ or $L_{2,1}(n) = 0$:

$$E[V(\underline{L}(n+1)) - V(\underline{L}(n)) \mid \underline{L}(n), A] < 0 \tag{9}$$

whereas $E[V(\underline{L}(n+1)) - V(\underline{L}(n)) \mid \underline{L}(n), B]$ maybe greater than 0 .

Finally, conditioned on $L_{1,1}(n) = 0$:

$$E[V(\underline{L}(n+1)) - V(\underline{L}(n)) \mid \underline{L}(n), B] < 0 \tag{10}$$

whereas $E[V(\underline{L}(n+1)) - V(\underline{L}(n)) \mid \underline{L}(n), A] > 0$.

1.4 Stable Algorithms

The value of Q above enables us to define the following algorithm that will be stable under all admissible traffic with i.i.d. arrivals for a 2x2 switch with three active flows:

1. If $L_{1,1}(n) = 0$, set crossbar to configuration *B*.
2. Else, if either $L_{1,2}(n) = 0$ or $L_{2,1}(n) = 0$, set crossbar to configuration *A*.
3. Else, set crossbar configuration to either *A* or *B*.

2 Relative Queue Sizes (Theorem 4.3)

In this section we prove that if or any arrival process to a 2x2 switch, if for some n

$$\left| \{L_{1,1}(n) + L_{2,2}(n)\} - \{L_{1,2}(n) + L_{2,1}(n)\} \right| \le 3 \tag{11}$$

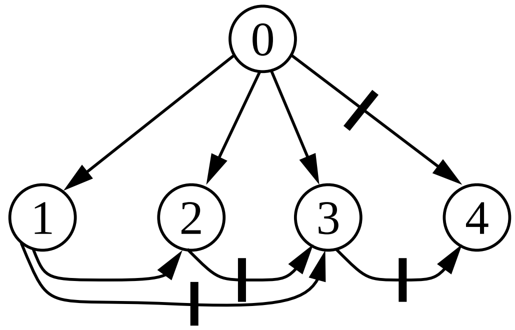


FIGURE A3.1 All possible single-step increases in $|D(n)|$. Arrows marked: ⟶ require two arrivals, which means that both queues in $L_M(n)$ are non-empty in the next cell time.

then for all $n' \geq n$

$$\left| \{L_{1,1}(n') + L_{2,2}(n')\} - \{L_{1,2}(n') + L_{2,1}(n')\} \right| \leq 4 . \tag{12}$$

For convenience, define

$$L_A(n) = L_{1,1}(n) + L_{2,2}(n) \quad , \qquad L_B(n) = L_{1,2}(n) + L_{2,1}(n) \tag{13}$$

and assume without loss of generality that $L_A(n) \geq L_B(n)$, i.e.

$$L_A(n) = L_B(n) + D(n) \tag{14}$$

for some $D(n) \geq 0$.

Finally, define

$$L_M(n) = \max(L_A(n), L_B(n)) \tag{15}$$

$$L_m(n) = \begin{cases} L_A(n) & \text{if } L_M(n) = L_B(n) \\ L_B(n) & \text{if } L_M(n) = L_A(n) \end{cases} . \tag{16}$$

Theorem A3.1: *All possible single step increases in* $|D(n)|$ *are shown in Figure 3.1.*

Proof:

Case (i): $D(n) = 0$. First we consider all possible values of $|D(n+1)|$ when $D(n) = 0$, i.e. $L_A(n) = L_B(n)$. We shall assume, without loss of generality that the two queues that contribute to $L_A(n)$ are served at time *n*.

At most two cells can arrive to the switch in a cell time, which means that

$$L_B(n+1) = L_B(n) + \begin{cases} 0 & : 0 \text{ arrivals} \\ 1 & : 1 \text{ arrival} \\ 2 & : 2 \text{ arrivals} \end{cases} \tag{17}$$

and that at most two cells can depart from the switch in a cell time, which means that

$$L_A(n+1) = L_A(n) + \begin{cases} -2 & : \text{(2 dep. and 0 arr.),} \\ -1 & : \text{(2 dep. and 1 arr.), or (1 dep and 0 arr.)} \\ 0 & : \text{(2 dep. and 2 arr.), or (1 dep and 1 arr.)} \\ 1 & : \text{(1 dep. and 2 arr.), or (0 dep and 1 arr.)} \end{cases} . \tag{18}$$

Note that there can only be 0 departures if $L_A(n) = 0$ and only 1 departure if either $L_{1,1}(n) = 0$ or $L_{2,2}(n) = 0$.

From Eq. 17 and Eq. 18 we find the following possible increases $D(n)$ when $D(n) = 0$.

$$|D(n+1)| = \begin{cases} 1: & (L_A(n) \to L_A(n) - 1),\ (L_B(n) \to L_B(n)) \\ 2: & (L_A(n) \to L_A(n) - 1),\ (L_B(n) \to L_B(n) + 1) \\ 3: & (L_A(n) \to L_A(n) - 2),\ (L_B(n) \to L_B(n) + 1) \\ 4: & (L_A(n) \to L_A(n) - 2),\ (L_B(n) \to L_B(n) + 2) \end{cases} \tag{19}$$

Note that $|D(n)| = 4$ if and only if two arrivals occur. This means that both queues that contribute to $L_M(n+1)$ are non-empty.

Case (ii)-(iv): $|D(n)| = 1, 2, 3$. By enumerating all transitions, as for $|D(n)| = 0$, we find the transitions from 1, 2, 3 in Figure 3.1.

Case (v): $|D(n)| = 4$. In cases (i)-(iv) we found that transitions into $|D(n)| = 4$ require that two arrivals occur and that both of the queues that contribute to $L_M(n)$ when $|D(n)| = 4$ are non-empty. As a result, the matching at time *n* serves two queues, hence

$$L_M(n+1) \le L_M(n)\,. \tag{20}$$

$L_m(n)$ is not served and so cannot decrease, therefore,

$$L_m(n+1) \ge L_m(n)\,. \tag{21}$$

Finally,

$$|D(n+1)| \le 4 \tag{22}$$

❐.

The transitions in Figure 3.1 indicate

1. For any queue occupancy such that $|D(n)| \le 3$, the next state is bounded by $|D(n+1)| \le 4$.
2. If $|D(n+1)| = 4$, then both queues in $L_M(n+1)$ are non-empty.
3. If both queues in $L_M(n)$ are non-empty and $|D(n)| = 4$, then $|D(n+1)| \le 4$.

Hence, if for some n, $|D(n)| \le 3$, then for all $n' \ge n$, $|D(n')| \le 4$ which proves the theorem.

APPENDIX 4

Stability of NxN Switch with i.i.d. Arrivals

1 Definitions

In this appendix we use the following definitions for an NxN switch:

1. The state vector, representing the occupancy of each queue at time n:

$$\underline{L}(n) \equiv (L_{1,1}(n), \ldots, L_{1,N}(n), \ldots, L_{N,1}(n), \ldots, L_{N,N}(n)) \; . \tag{1}$$

2. The (constant) arrival rate matrix:

$$\Lambda \equiv [\lambda_{i,j}] , \quad \text{where:} \quad \sum_{i=1}^{N} \lambda_{i,j} \le 1, \; \sum_{j=1}^{N} \lambda_{i,j} \le 1, \; \lambda_{i,j} \ge 0 \tag{2}$$

   and associated rate vector:

$$\underline{\lambda} \equiv (\lambda_{1,1}, \ldots, \lambda_{1,N}, \ldots, \lambda_{N,1}, \ldots, \lambda_{N,N}) \; . \tag{3}$$

3. The arrival matrix, representing the sequence of arrivals into each queue:

$$\mathbf{A}(n) \equiv [A_{i,j}(n)] , \quad \text{where:} \quad A_{i,j}(n) \equiv \begin{cases} 1 & \text{if arrival occurs at } Q(i,j) \text{ at time } n \\ 0 & \text{else} \end{cases} \tag{4}$$

   and associated arrival vector:

$$\underline{A}(n) \equiv (A_{1,1}(n), \ldots, A_{1,N}(n), \ldots, A_{N,1}(n), \ldots, A_{N,N}(n)). \tag{5}$$

4. The service matrix, indicating which queues are served at time n:

$$\mathbf{S}(n) \equiv [S_{i,j}(n)], \quad \text{where: } S_{i,j}(n) = \begin{cases} 1 & \text{if } Q(i,j) \text{ is served at time } n \\ 0 & \text{else} \end{cases} \tag{6}$$

and $\mathbf{S}(n) \in \mathbf{S}$, the set of service matrices.

Note that: $\sum_{i=1}^{N} S_{i,j}(n) = \sum_{j=1}^{N} S_{i,j}(n) = 1$

and hence $\mathbf{S}(n) \in \mathbf{S}$ is a *permutation matrix*.

We define the associated service vector:

$$\underline{S}(n) \equiv (S_{1,1}(n), \ldots, S_{1,N}(n), \ldots, S_{N,1}(n), \ldots, S_{N,N}(n)), \tag{7}$$

hence $\|\underline{S}(n)\|^2 = N$.

5. The *approximate* next-state vector:

$$\tilde{\underline{L}}(n+1) \equiv \underline{L}(n) - \underline{S}(n) + \underline{A}(n) , \tag{8}$$

which approximates the exact next-state of each queue

$$L_{i,j}(n+1) = [L_{i,j}(n) - S_{i,j}(n)]^{+} + A_{i,j}(n). \tag{9}$$

2 Main Theorem

Theorem A4.1: *An NxN switch is stable for the LQF algorithm under i.i.d. arrivals.*

3 Proof

Before proving this theorem, we first prove the following theorems.

Theorem A4.2: *The doubly stochastic matrices,* Λ*, form a convex set,* C*, with the set of extreme points equal to permutation matrices,* $\mathbf{S}$.

Proof: The set C is clearly convex: for all rate matrices $\Lambda_1, \Lambda_2 \in C$ and for every real number α, $0 < \alpha < 1$, the point $\alpha\Lambda_1 + (1-\alpha)\Lambda_2 \in C$. A permutation matrix S is doubly stochastic and is therefore a member of the set C. Furthermore, there are no two distinct matrices $\Lambda_1, \Lambda_2 \in C$ such that $\alpha\Lambda_1 + (1-\alpha)\Lambda_2 = S$, for real α, $0 < \alpha < 1$. Hence, a permutation matrix is an extreme point of C. ❐

Theorem A4.3: $\underline{L}^T(n)\underline{\lambda} - \underline{S}^*(n) \le 0, \quad \forall\, (\underline{L}(n), \underline{\lambda})$*, where* $\underline{S}^*(n) = \max(\underline{L}^T(n)\underline{S}(n))$*, the service matrix selected by the LQF algorithm to maximize* $\underline{L}^T(n)\underline{S}(n)$.

Proof: Consider the linear programming problem:

$$\begin{aligned} &\max(\underline{L}^T(n)\underline{\lambda}) \\ &\text{s.t.} \quad \sum_{i=1}^{N} \lambda_{i,j} \le 1, \ \sum_{j=1}^{N} \lambda_{i,j} \le 1, \ \lambda_{i,j} \ge 0 \end{aligned} \tag{10}$$

which has a solution equal to an extreme point of the convex set, C. Hence,

$$\max(\underline{L}^T(n)\underline{\lambda}) \le \max(\underline{L}^T(n)\underline{S}(n)) \tag{11}$$

and so $\underline{L}^T(n)\underline{\lambda} - \max(\underline{L}^T(n)\underline{S}(n)) \le 0$. ❐

Theorem A4.4: $E[\underline{\tilde{L}}^T(n+1)\underline{\tilde{L}}(n+1) - \underline{L}^T(n)\underline{L}(n) | \underline{L}(n)] \le 2N, \quad \forall\, \underline{\lambda}$.

Proof:

$$\begin{aligned} &\underline{\tilde{L}}^T(n+1)\underline{\tilde{L}}(n+1) - \underline{L}^T(n)\underline{L}(n) \\ &= (\underline{L}(n) - \underline{S}(n) + \underline{A}(n))^T (\underline{L}(n) - \underline{S}(n) + \underline{A}(n)) - \underline{L}^T(n)\underline{L}(n) \\ &= 2\underline{L}^T(n)(\underline{A}(n) - \underline{S}(n)) + (\underline{S}(n) - \underline{A}(n))^T (\underline{S}(n) - \underline{A}(n)) \\ &= 2\underline{L}^T(n)(\underline{A}(n) - \underline{S}(n)) + k, \end{aligned} \tag{12}$$

where $0 \le k \le 2N$. $k \ge 0$ because $\underline{S}(n) - \underline{A}(n)$ is a real vector, and $k \le 2N$ because $\|\underline{S}(n) - \underline{A}(n)\|^2 \le 2N$.

Taking the expected value:

$$E[\tilde{\underline{L}}^T(n+1)\tilde{\underline{L}}(n+1) - \underline{L}^T(n)\underline{L}(n) \mid \underline{L}(n)] \le E[2\underline{L}^T(n)(\underline{A}(n) - \underline{S}(n))] + 2N \\ = 2\underline{L}^T(n)(\underline{\lambda} - \underline{S}^*(n)) + 2N. \tag{13}$$

From Theorem 4.4 we know that $2\underline{L}^T(n)(\underline{\lambda} - \underline{S}^*(n)) \le 0$, proving the theorem. ❐

Theorem A4.5: $\mathrm{E}[\tilde{\underline{L}}^T(n+1)\tilde{\underline{L}}(n+1) - \underline{L}^T(n)\underline{L}(n) \mid \underline{L}(n)] \le -\varepsilon\|\underline{L}(n)\| + 2N$, $\varepsilon > 0$,

$\forall \underline{\lambda} \le (1-\beta)\underline{\lambda}_m$, $0 < \beta < 1$, *where* $\underline{\lambda}_m$ *is any rate vector such that* $\|\underline{\lambda}_m\|^2 = N$.

Proof:

$$\underline{L}^T(n)(\underline{\lambda} - \underline{S}^*(n)) \le \underline{L}^T(n)\{\underline{\lambda}_m - \underline{S}^*(n)\} - \underline{L}^T(n)(\beta\underline{\lambda}_m) \\ \le 0 - \beta\|\underline{L}(n)\| \cdot \|\underline{\lambda}_m\| \cos\theta \tag{14}$$

where θ is the angle between $\underline{L}(n)$ and $\underline{\lambda}_m$.

We now show that $\cos\theta > \delta$ for some $\delta > 0$ whenever $\underline{L}(n) \ne \underline{0}$. First, we show that $\cos\theta > 0$. We do this by contradiction: suppose that $\cos\theta = 0$, i.e. $\underline{L}(n)$ and $\underline{\lambda}_m$ are orthogonal. This can only occur if $\underline{L}(n) = \underline{0}$, or if for some i, j, both $\lambda_{i,j} = 0$ and $L_{i,j}(n) > 0$, which is not possible: for arrivals to have occurred at queue $Q(i,j)$, $\lambda_{i,j}$ must be greater than zero. Therefore, $\cos\theta > 0$ unless $\underline{L}(n) = \underline{0}$. Now we show that $\cos\theta$ is bounded away from zero, i.e. that $\cos\theta > \delta$ for some $\delta > 0$. Because $\lambda_{i,j} > 0$ wherever $L_{i,j}(n) > 0$, and because $\|\underline{\lambda}\| \le \sqrt{N}$,

$$\cos\theta = \frac{\underline{L}^T(n)\underline{\lambda}}{\|\underline{L}(n)\|\|\underline{\lambda}\|} \ge \frac{L_{max}(n)\lambda_{min}}{\|\underline{L}(n)\|\sqrt{N}}, \tag{15}$$

where $\lambda_{min} = \min(\lambda_{i,j}, 1 \le i, j \le N)$ and $L_{max}(n) = \max(L_{i,j}(n), 1 \le i, j \le N)$. Also,

$$\|\underline{L}(n)\| \le [N^2 L_{max}^2(n)]^{1/2} = N L_{max}(n), \tag{16}$$

and so $\cos\theta$ is bounded by

$$\cos\theta \ge \frac{\lambda_{min}}{N\sqrt{N}} \tag{17}$$

Therefore

$$E\left[\underline{\tilde{L}}^T(n+1)\underline{\tilde{L}}(n+1) - \underline{L}^T(n)\underline{L}(n) \middle| \underline{L}(n)\right] \le -\frac{\beta\lambda_{min}}{\sqrt{N}}\|\underline{L}(n)\| + 2N. \ \square \tag{18}$$

Theorem A4.6: $E\left[\underline{L}^T(n+1)\underline{L}(n+1) - \underline{L}^T(n)\underline{L}(n) \middle| \underline{L}(n)\right] \le -\varepsilon\|\underline{L}(n)\| + N^2 + 2N$, $\varepsilon > 0$

$\forall\ \underline{\lambda} \le (1-\beta)\,\underline{\lambda}_m(n), \quad 0 < \beta < 1$.

Proof:

$$L_{i,j}(n+1) = \tilde{L}_{i,j}(n+1) + \begin{cases} 1 & \text{if } L_{i,j}(n) = 0, S_{i,j}(n) = 1 \\ 0 & \text{else} \end{cases}, \tag{19}$$

therefore

$$\underline{L}^T(n+1)\underline{L}(n+1) - \underline{\tilde{L}}^T(n+1)\underline{\tilde{L}}(n+1) \le N^2, \tag{20}$$

and so

$$E\left[\underline{L}^T(n+1)\underline{L}(n+1) - \underline{L}^T(n)\underline{L}(n) \middle| \underline{L}(n)\right] \le E\left[\underline{\tilde{L}}^T(n+1)\underline{\tilde{L}}(n+1) - \underline{L}^T(n)\underline{L}(n) \middle| \underline{L}(n)\right] + N^2. \tag{21}$$

Using Theorem 4.5 this concludes the proof. ❐

Theorem A4.7: *There exists a* $V(\underline{L}(n))$ *s.t.* $E[V(\underline{L}(n+1)) - V(\underline{L}(n)) | \underline{L}(n)] \le -\varepsilon\|\underline{L}(n)\| + k$, *where* $k, \varepsilon > 0$.

Proof: $V(\underline{L}(n)) = \underline{L}^T(n)\underline{L}(n)$ and $k = N^2 + 2N$ in Theorem 4.6.❐

We are now ready to prove the main theorem.

Proof of Main Theorem: $V(\underline{L}(n))$ in Theorem 4.7 is a quadratic Lyapunov function and, according to the argument of Kumar and Meyn [27], it follows that the switch is stable. ❐