What, you’ve been working on the same problem too?"

— Conversation with Devavrat Shah†

Proofs for Chapter 9

Definition I.1. Domination: Let $v = (v_1, v_2, \ldots, v_N)$, and $u = (u_1, u_2, \ldots, u_N)$ denote the values of $C(i, t)$ for two different systems of $N$ counters at any time $t$. Let $\pi, \sigma$ be an ordering of the counters $(1, 2, 3, \ldots N)$ such that they are in descending order, i.e., for $v$ we have, $v_{\pi(1)} \geq v_{\pi(2)} \geq v_{\pi(3)} \geq \cdots \geq v_{\pi(N)}$ and for $u$ we have $u_{\sigma(1)} \geq u_{\sigma(2)} \geq u_{\sigma(3)} \geq \cdots \geq u_{\sigma(N)}$.

We say that $v$ dominates $u$ denoted $v \gg u$, if $v_{\pi(i)} \geq u_{\sigma(i)}, \forall i$. Every arrival can possibly increment any of $N$ different counters. The set of all possible arrival patterns at time $t$ can be defined as: $\Omega_t = \{(w_1, w_2, w_3, \ldots, w_t), 1 \geq w_i \geq N, \forall i\}$.

Theorem I.1. (Optimality of LCF-CMA). Under arrival sequence $a(t) = (a_1, a_2, a_3, \ldots, a_t)$, let $q(a(t), P_c) = (q_1, q_2, q_3, \ldots, q_N)$ denote the count $C(i, t)$ of $N$ counters at time $t$ under service policy $P_c$. For any service policy $P$, there exists a 1–1 function $f_{P,LCF}^t : (\Omega_t \rightarrow \Omega_t)$, for any $t$ such that $q(f_{P,LCF(w)}^t, P) \gg q(w, LCF), \forall(w \in \Omega_t), \forall t$.

Proof. We prove the existence of such a function $f_{P,LCF}^t$ inductively over time $t$. Let us denote the counters of the LCF system by $(l_1, l_2, l_3, \ldots, l_N)$ and the counters of the $P$ system by $(p_1, p_2, p_3, \ldots, p_N)$. It is trivial to check that there exists such a function

†”Might as well submit a joint paper then!”, Stanford University, 2001.
for $t = 1$. Inductively assume that $f_{P,LCF}^t$ exists with the desired property until time $t$, and we want to extend it to time $t + 1$. This means that there exists ordering $\pi^t, \sigma^t$ such that, $l_{\pi^t(i)} \leq p_{\sigma^t(i)}, \forall i$. Now, at the time $t + 1$, a counter may be incremented and a counter may be completely served. We consider both these parts separately below:

**Part 1: (Arrival)** Let a counter be incremented at time $t + 1$ in both systems. Suppose that counter $\pi^t(k)$ is incremented in the LCF system. Then extend $f_{P,LCF}^t$ for $t + 1$ by letting an arrival occur in counter $\sigma^t(k)$ for the $P$ system. By induction, we have $l_{\pi^t(i)} \leq p_{\sigma^t(i)}, \forall i$. Let $\pi^{t+1}, \sigma^{t+1}$ be the new ordering of the counters of the LCF and $P$ systems respectively. Since one arrival occurred to both the systems in a queue with the same relative order, the domination relation does not change.

**Part 2: (Service)** Let one of the counters be served at time $t + 1$. Under the LCF policy, the counter $\pi^t(1)$ with count $l_{\pi^t(1)}$ will be served and its count is set to zero, i.e., $C(\pi^t(1), t + 1) = 0$, while under $P$ any queue can be served out, depending on the CMA prescribed by $P$. Let $P$ serve the counter with rank $k$, i.e., counter $\sigma^t(k)$. Then we can create a new ordering $\pi^{t+1}, \sigma^{t+1}$ as follows:

\[
\pi^{t+1}(i) = \pi^t(i + 1), \quad 1 \leq i \leq N - 1, \quad \pi^{t+1}(N) = \pi^t(1). \tag{I.1}
\]

\[
\sigma^{t+1}(i) = \sigma^t(i), \quad 1 \leq i \leq k - 1, \quad \sigma^{t+1}(i) = \sigma^t(i + 1), \quad k \leq i \leq N - 1, \quad \sigma^{t+1}(N) = \sigma^t(k). \tag{I.2}
\]

Under this definition, it is easy to check that, $l_{\pi^{t+1}(i)} \leq p_{\sigma^{t+1}(i)}, \forall i$ given $l_{\pi^t(i)} \leq p_{\sigma^t(i)}, \forall i$. Thus we have shown explicitly how we can extend to $f_{P,LCF}^t$ to $f_{P,LCF}^{t+1}$ with the desired property. Hence it follows inductively that LCF is dominated by any other policy $P$. $\square$