"What, you've been working on the same problem too?".

— Conversation with Devavrat Shah†

Proofs for Chapter 9

ℵ**Definition I.1. Domination:** Let $v = (v_1, v_2, \ldots, v_N)$, and $u = (u_1, u_2, \ldots, u_N)$ denote the values of $C(i,t)$ for two different systems of N counters at any time t. Let π, σ be an ordering of the counters $(1, 2, 3, \ldots N)$ such that they are in descending order, *i.e.*, for v we have, $v_{\pi(1)} \geqslant v_{\pi(2)} \geqslant v_{\pi(3)} \geqslant \cdots \geqslant v_{\pi(N)}$ and for u we have $u_{\sigma(1)} \geqslant u_{\sigma(2)} \geqslant u_{\sigma(3)} \geqslant \cdots \geqslant u_{\sigma(N)}$.

We say that v dominates u denoted $v \ggg u$, if $v_{\pi(i)} \geqslant u_{\sigma(i)}, \forall i$. Every arrival can possibly increment any of N different counters. The set of all possible arrival patterns at time t can be defined as: $\Omega_t = \{(w_1, w_2, w_3, \ldots, w_t), 1 \geqslant w_i \geqslant N, \forall i\}$.

Theorem I.1. *(Optimality of LCF-CMA). Under arrival sequence* $a(t) = (a_1, a_2, a_3, \ldots, a_t)$*, let* $q\,(a(t), P_c) = (q_1, q_2, q_3, \ldots, q_N)$ *denote the count* $C(i,t)$ *of* N *counters at time* t *under service policy* P_c*. For any service policy* P*, there exists a* $1-1$ *function* $f^t_{P,LCF} : (\Omega_t \rightarrow \Omega_t)$*, for any* t *such that* $q(f^t_{P,LCF(w),P}) \ggg q(w, LCF), \forall (w \in \Omega_t), \forall t.$

Proof. We prove the existence of such a function $f^t_{P,LCF}$ inductively over time t. Let us denote the counters of the LCF system by $(l_1, l_2, l_3, \ldots, l_N)$ and the counters of the P system by $(p_1, p_2, p_3, \ldots, p_N)$. It is trivial to check that there exists such a function

†"Might as well submit a joint paper then!", Stanford University, 2001.

for $t = 1$. Inductively assume that $f^t_{P,LCF}$ exists with the desired property until time t, and we want to extend it to time $t+1$. This means that there exists ordering π^t, σ^t such that, $l_{\pi^t(i)} \leqslant p_{\sigma^t(i)}, \forall i$. Now, at the time $t+1$, a counter may be incremented and a counter may be completely served. We consider both these parts separately below:

- ***Part 1*****: (Arrival)** Let a counter be incremented at time $t+1$ in both systems. Suppose that counter $\pi^t(k)$ is incremented in the LCF system. Then extend $f^t_{P,LCF}$ for $t+1$ by letting an arrival occur in counter $\sigma^t(k)$ for the P system. By induction, we have $l_{\pi^t(i)} \leqslant p_{\sigma^t(i)}, \forall i$. Let π^{t+1}, σ^{t+1} be the new ordering of the counters of the LCF and P systems respectively. Since one arrival occurred to both the systems in a queue with the same relative order, the domination relation does not change.
- ***Part 2*****: (Service)** Let one of the counters be served at time $t+1$. Under the LCF policy, the counter $\pi^t(1)$ with count $l_{\pi^t(1)}$ will be served and its count is set to zero, *i.e.*, $C(\pi^t(1), t+1) = 0$, while under P any queue can be served out, depending on the CMA prescribed by P. Let P serve the counter with rank k, *i.e.*, counter $\sigma^t(k)$. Then we can create a new ordering π^{t+1}, σ^{t+1} as follows:

$$\pi^{t+1}(i) = \pi^t(i+1), \quad 1 \leqslant i \leqslant N-1, \quad \pi^{t+1}(N) = \pi^t(1). \tag{I.1}$$

$$\begin{aligned} \sigma^{t+1}(i) = \sigma^t(i), \quad 1 \leqslant i \leqslant k-1, \\ \sigma^{t+1}(i) = \sigma^t(i+1), \quad k \leqslant i \leqslant N-1, \quad \sigma^{t+1}(N) = \sigma^t(k). \end{aligned} \tag{I.2}$$

Under this definition, it is easy to check that, $l_{\pi^{t+1}(i)} \leqslant p_{\sigma^{t+1}(i)}, \forall i$ given $l_{\pi^t(i)} \leqslant p_{\sigma^t(i)}, \forall i$. Thus we have shown explicitly how we can extend to $f^t_{P,LCF}$ to $f^{t+1}_{P,LCF}$ with the desired property. Hence it follows inductively that LCF is dominated by any other policy P.❐