Characterization of Networks Supporting Multi-dimensional Linear Interval Routing Schemes*

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Abstract

An *Interval Routing Scheme (IRS)* is a well-known, space efficient routing strategy for routing messages in a distributed network. In this scheme, each node of the network is assigned an integer label and each link at each node is labeled with an interval. The interval assigned to a link e at a node v indicates the set of destination addresses of the messages which should be forwarded through e at v. A *Multi-dimensional Interval Routing Scheme (MIRS)* is a generalization of IRS in which each node is assigned a multi-dimensional label (which is a list of d integers for the d-dimensional case). The labels assigned to the links of the network are also multi-dimensional (a list of d 1-dimensional intervals). The class of networks supporting linear IRS (in which the intervals are not cyclic) is already known for the 1-dimensional case [FG94]. In this paper, we generalize this result and completely characterize the class of networks supporting linear MIRS (or MLIRS) for a given number of dimensions d. We show that by increasing d, the class of networks supporting MLIRS is strictly expanded. We also give a characterization of the class of networks supporting strict MLIRS (which is an MLIRS in which the intervals assigned to the links incident to a node v, does not contain the label of v).

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1 Introduction

One of the most fundamental tasks in any network of computers is routing messages between pairs of nodes. The classical method used for routing messages in a network is to store a *routing table* at each node of the network. A routing table has one entry for each destination address which indicates which of the adjacent links should be used to forward the message towards that destination.

Each routing table requires $O(n)$ space in an n-node network, which is not efficient (and even feasible) for large networks of computers. The methods to reduce the amount of space needed at each node have been intensively studied and there are many techniques to compress the size of routing tables [FJ88, FJ89, ABNLP90, TvL95]. The general idea is to group the destination addresses that correspond to the same outgoing link (at a node), and to encode the group so that it is easy to verify if a given destination address is in the group or not. A well-known solution is to use intervals as groups of destination addresses.

In an *Interval Routing Scheme (IRS)*, which was originally introduced by Santoro and Khatib [SK85], each node of the network is assigned an integer label taken from $\{1, 2, ..., n\}$ and each link of the network at each node is assigned an interval which can be cyclic. Routing messages is completed in a distributed way. At each intermediate node v, if the label of the node equals the destination address, *dest*, the routing process ends. Otherwise, the message is forwarded through a link labeled by an interval I, such that $dest \in I$. Clearly, this method requires $O(l)$ space at each node (l is the number of links at the node), which is an efficient memory allocation.

A *Linear Interval Routing Scheme (LIRS)* is an IRS in which the intervals are not cyclic. The concept of LIRS was first introduced by Bakker et al. [BvLT91]. They mentioned practical reasons for which we allow only the use of *linear* intervals and not *cyclic* ones. This notions is especially useful to derive results on networks built by cartexian products (as hypercubes and torus) [FG98]. Also, a *Strict Interval Routing Scheme (SIRS)* is an IRS in which the interval assigned to a link e at a node v does not contain the label of v. A *Strict and Linear Interval Routing Scheme (SLIRS)* is an IRS which is both linear and strict. If we assign k intervals to each link of the network we will have a k-IRS (respectively, k-LIRS, k-SIRS, and k-SLIRS). Gavoille has done a survey of results concerning this method [Gav00].

It has been proved that any network supports an SIRS and therefore an IRS [SK85, vLT87]. The class of networks which support LIRS and SLIRS have also been characterized by Fraigniaud and Gavoille which excludes a large class of networks [FG94]. They define a class of graphs called *lithium graphs* and show that a network supports an LIRS if and only if its underlying graph is not a lithium graph. They also show that a network supports an SLIRS if and only if its underlying graph is not a *weak lithium graph*.

A very interesting extension of IRS is a *Multi-dimensional Interval Routing Scheme (MIRS)* in which the labels assigned to the nodes are elements from $\mathbb{N}^d$ (in the d-dimensional case) and each link is labeled with a d-tuple $([a_1, b_1], [a_2, b_2], ..., [a_d, b_d])$ of intervals, $a_i, b_i \in \mathbb{N}$, for $1 \leqslant i \leqslant d$ [FGNT98]. The routing process in an MIRS is quite similar to the routing process in 1-dimensional IRS.

A network is said *to be in* $\langle k, d\rangle$*-MIRS* or *support* $\langle k, d\rangle$*-MIRS* if there is a d-dimensional MIRS with k intervals in each link such that for any pair of nodes s and t, the message originating from s eventually reaches t. The classes $\langle k, d\rangle$-MLIRS and $\langle k, d\rangle$-MSLIRS are defined similarly. The only known classes of networks which support different variations of MIRS are specific interconnection networks such as rings, grids, tori, hypercubes and chordal rings. In this paper, we will investigate the problem of characterizing classes of networks supporting MIRS. We give a complete characterization of the class of networks supporting $\langle 1, d\rangle$-MLIRS and $\langle 1, d\rangle$-MSLIRS. We show that the class of networks supporting $\langle 1, d\rangle$-MLIRS ($\langle 1, d\rangle$-MSLIRS) is a strict subset of the class of networks supporting $\langle 1, d+1\rangle$-MLIRS ($\langle 1, d+1\rangle$-MSLIRS) and therefore, increasing the number of dimensions in an MLIRS (MSLIRS) increases the power of the routing scheme.

The rest of this paper is organized as follows: first, we will introduce some definitions and preliminaries in Section 2. In Section 3 we will characterize the class of graphs supporting $\langle 1, d\rangle$-MLIRS. Then, in Section 4, based on the arguments of the previous section, we will give a characterization for graphs supporting $\langle 1, d\rangle$-MSLIRS. Finally, in Section 5 we will conclude and give a list of open problems.

2 Preliminaries

Throughout this paper, a network is modeled by a graph $G = (V, E)$. The set V of vertices of the graph represents nodes in the network and the set E of edges represents the links between the nodes in the network. We assume that the graph is simple and does not have any self-loops. For any edge $(u, v) \in E$ we will use both (u, v) and (v, u) in order to assign two unidirectional labels to the edge, but the edge is assumed to be undirected. We refer the reader to standard texts for basic graph theoretic definitions [BM76, Wes96].

A graph G is said to be *connected* if for any pair of vertices, s and t, there is a path connecting s and t. In this paper, we always assume that the network is connected. If removing an edge e disconnects a graph G, e is called a *bridge*. If a graph does not have a bridge, it is said to be *edge-biconnected*. *Edge-biconnected components of a graph* G are maximal subgraphs of G which are edge-biconnected.

Observation 1. *If G_1 and G_2 are two edge-biconnected components in a graph G, then any path P connecting G_1 and G_2 goes through a unique bridge connected to G_1.*

In the following section, we will give a characterization for the class of networks supporting a $\langle 1, d\rangle$-MLIRS.

3 Characterization of networks supporting $\langle 1, d\rangle$-MLIRS

In this section we first give some examples of graphs which do not support $\langle 1, d\rangle$-MLIRS. Using the idea behind these examples, we introduce a class of graphs which do not support $\langle 1, d\rangle$-MLIRS. Finally, we show that for any graph that is not in this class, one can always construct a $\langle 1, d\rangle$-MLIRS.

Bakker et al. [BvLT91] have proved that the graph shown in Figure 1 (a) (known as the Y graph) does not have an LIRS (which is a $\langle 1, 1\rangle$-MLIRS). Here, we prove a similar result in the d-dimensional case. First, let us start by generalizing the definition of a Y graph.

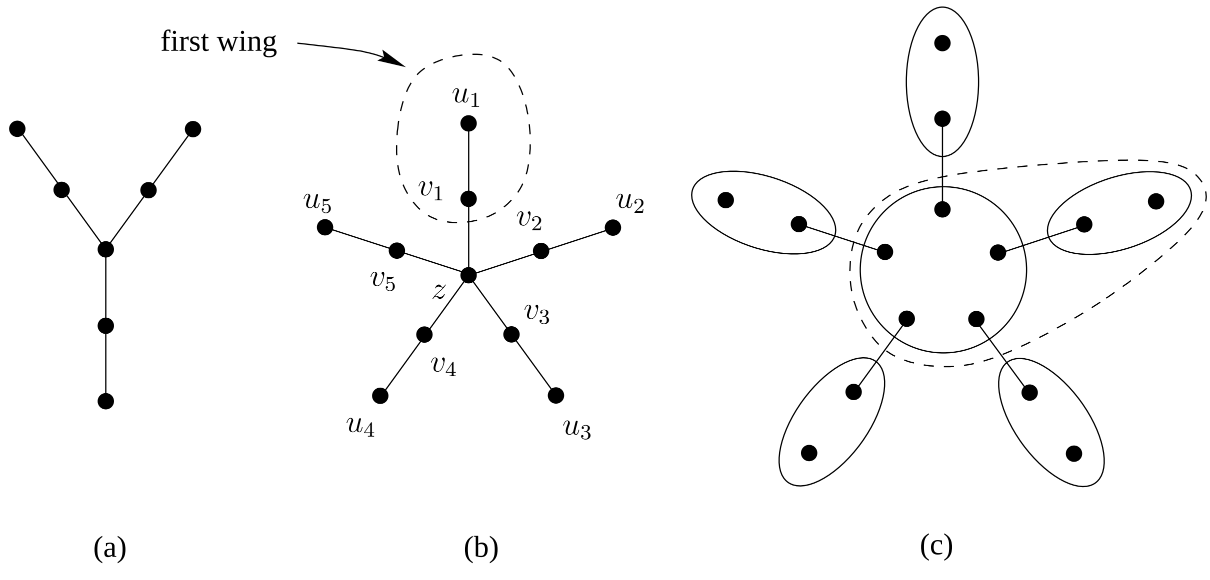

Figure 1: (a) The Y graph (b) The Y_5 graph (c) A 5-windmill graph.

Definition 1. The Y_k *graph* is a graph having $2k + 1$ vertices $u_1, u_2, ..., u_k$, $v_1, v_2, ...,$ v_k and z. There is an edge connecting u_i to v_i, for every i, $1 \leqslant i \leqslant k$, and another edge connecting each v_i to z, $1 \leqslant i \leqslant k$ (Figure 1 (b)). We call the subgraph consisting of u_i and v_i the *ith wing* of the graph.

The Y graph of Figure 1 (a) is a Y_3 graph by our new definition. To prove that the Y_3 graph does not have an LIRS let us assume it has an LIRS and the vertices of the graph are assigned integer labels taken from $\{1, 2, ..., 7\}$. Since we have three wings, there is a wing, say the ith wing, which does not contain 1 or 7 (the minimum or the maximum label). Now, the interval assigned to the edge (v_i, z) at v_i must contain both 1 and 7. Therefore, this interval contains the label of u_i which is not possible.

We can prove a similar result for d-dimensional LIRS and for the Y_{2d+1} graph. In fact, we can immediately observe that if each wing of the Y_{2d+1} graph had more than just two vertices, as long as those vertices are not directly connected to the vertex z or to the vertices in other wings, the graph cannot support a d-dimensional MLIRS. In order to prove this more general statement, we define a k-windmill graph as follows.

Definition 2. A k-*windmill* graph is a connected graph with $k + 1$ connected components (not necessarily maximal) $A_1, A_2, ..., A_k$ (*arms* of the k-windmill graph) and R (*center* of the k-windmill graph) such that:

(i) each component $A_i, 1 \leqslant i \leqslant k$, has at least two vertices;

(ii) there is no edge connecting A_i to A_j for $1 \leqslant i, j \leqslant k$ and $i \neq j$; and

(iii) each component $A_i, 1 \leqslant i \leqslant k$, is connected with R by exactly one bridge.

Figure 1 (c) illustrates a 5-windmill graph. Obviously, by this definition, a Y_k graph is also a k-windmill graph. Also, as Figure 1 (c) indicates, a k-windmill graph is a i-windmill graph for any i, $1 \leqslant i \leqslant k - 1$. This can easily be shown by expanding R to include $A_{i+1}, ..., A_k$.

Lemma 1. Any $(2d + 1)$-windmill graph $\notin \langle 1, d\rangle$-MLIRS.

Before proving this lemma, let us give a new definition, which will be used in the proof. We consider a set of points P in d-dimensional space. For any dimension i, $1 \leqslant i \leqslant d$, if the ith coordinate of a point b in P is less than or equal to the ith coordinate of every other point in P, b is called a *minimum point for the* i*th dimension*. A maximum point is defined similarly. A *boundary set* B *of* P is a minimal set of points in P containing a minimum and a maximum point for each dimension $i, 1 \leqslant i \leqslant d$, where one point can be both the minimum and the maximum point for the same or different dimensions.

Figure 2 illustrates an example of a boundary set in 2-dimensional space. Here, $P = \{1, \ldots, 7\}$ and $\{1, 5, 7\}$ is a boundary set of P. The set $\{2, 5, 7\}$ is also a boundary set of P. We note that point 7 is the maximum point for one dimension and the minimum point for another dimension.

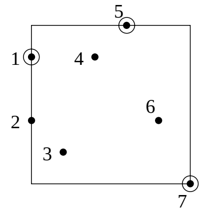

Figure 2: An example of a boundary set in 2-dimensional space.

For any set of points in d-dimensional space, the number of points in any boundary set is at most $2d$. It is easy to show that if an interval contains the points in the boundary set B of a set of points P, it contains all points in P. Now we can easily prove Lemma 1. In this proof, we consider the d-dimensional labels of vertices as points in d-dimensional space.

Proof. (Lemma 1) Let us assume, by way of contradiction, that there is a $\langle 1, d\rangle$-MLIRS for a given $(2d+1)$-windmill graph ($d \geqslant 1$) and consider the boundary set B of the vertices of the graph. We have at most $2d$ vertices in the boundary set B. Since a $(2d+1)$-windmill graph has $2d+1$ arms, there is an arm, say the jth arm, that does not contain any vertex in the boundary set B. Every d-dimensional interval containing all of the vertices in B contains all vertices of $(2d+1)$-windmill graph as well. Thus, the interval assigned to the bridge connecting the jth arm to the center of the $(2d+1)$-windmill graph, say (u, v) (u is in the jth arm and v is a vertex in the center of the graph) contains all vertices in the $(2d+1)$-windmill graph. The jth wing has at least another vertex other than u, say u'. Hence, the interval assigned to the edge (u, v) includes u'. Obviously, there is no path going through (u, v) to reach u', which is a contradiction. ■

Lemma 1 introduces a class of graphs which do not support $\langle 1, d\rangle$-MLIRS. In other words, it states a necessary condition for a graph to support a $\langle 1, d\rangle$-MLIRS. In the rest of this section we will show that this is also a sufficient condition.

Fraigniaud and Gavoille have proved that a graph supports LIRS if and only if it is not a lithium graph [FG94] (which is exactly the 3-windmill graph). We will use this result as the basis for an inductive construction of a $\langle 1, d\rangle$-MLIRS for a given graph G. We start with some new definitions.

Definition 3. In a graph G, a *chain of edge-biconnected components*, or a *chain* for short, is a set of edge-biconnected components of G with a special ordering of these edge-biconnected components, say $G_1, G_2, ..., G_k$, such that:

(i) for each $i, 1 \leqslant i \leqslant k-1$, there is a bridge connecting G_i to G_{i+1};

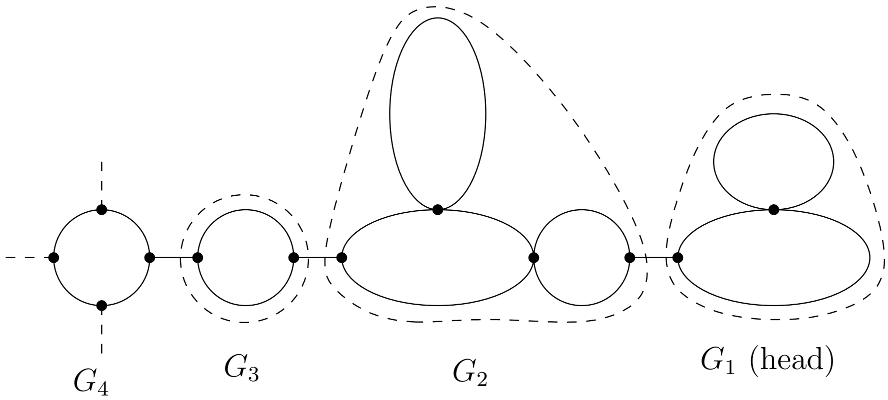

Figure 3: The dashed curves indicate edge-biconnected components in this figure. The edge-biconnected components G_1 and G_2 form a chain. The edge-biconnected components G_1, G_2 and G_3 (and not G_4) form a perfect chain.

(ii) G_1 is connected to exactly one bridge in G;

(iii) each edge-biconnected component G_i, $2 \leqslant i \leqslant k-1$ is connected to exactly two bridges in G; and

(iv) The edge-biconnected component G_k is connected to either one or two bridges.

We call G_1 the *head* and G_k the *tail* of the chain. Trivially if $k = 1$ then G_1 is both the head and the tail of the chain. A chain is said to be *perfect* if the tail of the chain is connected to an edge-biconnected component which is connected to more than two bridges.

3.1 Properties of chains and k-windmill graphs

In this section we review some of the properties of chains and k-windmill graphs. The first observation follows directly from the definition of a chain.

Observation 2. *A perfect chain in a graph G is a proper induced subgraph of G, and the tail of a perfect chain (which is an edge-biconnected component) is connected to the rest of the graph by a bridge.*

The edge-biconnected components G_1, G_2 and G_3 in the graph depicted in Figure 3 and the bridges connecting them form a chain. G_1 and G_3 are the head and the tail of this chain, respectively. This is also a perfect chain since G_3 (tail) is connected to an edge-biconnected component (G_4) which is connected to more than two bridges. As mentioned in Observation 2, G_3 (which is the tail of the perfect chain) is connected to the rest of the graph by a bridge. Since, G_3 is connected to exactly two bridges, the edge-biconnected components G_1 and G_2 does not form a perfect chain.

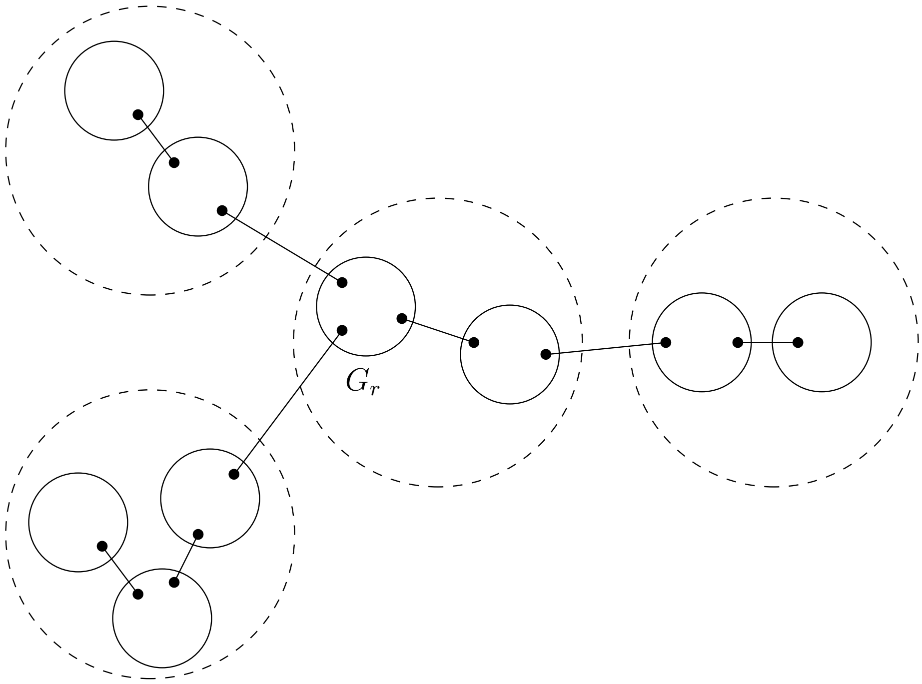

Figure 4: Edge-biconnected components in a 3-windmill graph.

Lemma 2. If a graph G is a k-windmill graph for $k \geqslant 3$ then it is not a chain.

Proof. We consider each edge-biconnected component of G as a super-node. Clearly, the resulting graph is a tree (otherwise, we have a cycle which contains some bridges, a contradiction). Since, G is a k-windmill graph ($k \geqslant 3$), there is a node v in this tree such that the degree of v is at least 3 (the super-node G_r in Figure 4). In any chain, each edge-biconnected component is connected to at most 2 other edge-biconnected components. Therefore, G is not a chain. ∎

Lemma 3. Any non-trivial (having at least one vertex) graph G which is not a chain contains a perfect chain as a proper induced subgraph.

Proof. Similar to the proof of Lemma 2, if we consider each edge-biconnected component of G as a super-node we will have a tree. Any tree has at least one leaf. The chain starting with this leaf and going to the nearest super-node with degree at least three is a perfect chain (since G is not a chain such a super-node always exists). ∎

For example, in the graph depicted in Figure 3 if we consider the chain starting from the super-node G_1 and going to G_3 (which is connected to G_4 which is of degree four) we have a perfect chain.

In constructing a $\langle 1, d\rangle$-MLIRS, we will use this lemma in the induction step to reduce the size of the graph. This reduction has a very nice property that is the heart of the main proof, which is stated in the following lemma.

Lemma 4. If a graph G is not a chain and is not a k-windmill graph ($k > 3$), we can remove any perfect chain from G and the resulting graph is not a $(k-1)$-windmill graph.

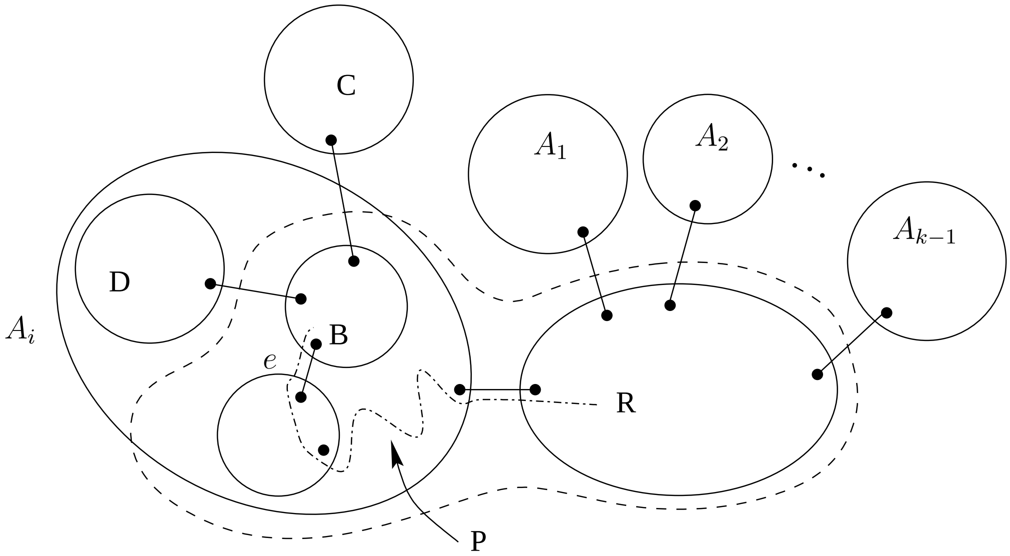

Figure 5: C and D will become arms in the k-windmill graph.

Proof. Since G is not a chain, by Lemma 3, there is a perfect chain C which is a proper induced subgraph of G. We let G' denote the graph $G - C$. We assume, to the contrary, that G' is a $(k-1)$-windmill graph. By the definition of a $(k-1)$-windmill graph, G' has k disjoint sets of vertices $A_1, A_2, ..., A_{k-1}$ and R. Since C is a perfect chain, by Observation 2 its tail is connected to G' by a bridge. C cannot be connected to R, otherwise G must be a k-windmill graph. Let us assume that C is connected to an edge-biconnected component, B, which is in the arm A_i for some i, $1 \leqslant i \leqslant k-1$ (Figure 5).

By the definition of a perfect chain, the edge-biconnected component B is connected to at least three bridges, one connecting B to C and at least two other bridges connecting B to some other edge-biconnected components in G'. By Observation 1 all the paths connecting B and R go through one of the bridges connected to B, say e. We let D be the edge-biconnected component which is connected to B and is not connected to e.

Now, we expand R to contain B and all the edge-biconnected components in the arm A_i except D. Since G is a $(k-1)$-windmill graph it has $k-2$ arms other than A_i. We can also consider C and D as two new arms. Hence, G has k arms and is a k-windmill graph, a contradiction. ∎

3.2 Characterization

In this section we will prove the main result of this paper. First, we need to show how to convert a d-dimensional IRS into a $(d+1)$-dimensional IRS.

If a graph G supports a $\langle 1, d\rangle$-MLIRS ($\langle 1, d\rangle$-MSLIRS), we can convert the d-dimensional to a $(d+1)$-dimensional one, by adding a new coordinate to the labels of vertices. The label of this coordinate is set to zero for all vertices. We also set the newly added coordinate of each interval to be $[0..0]$. It is a trivial task to verify that this IRS routes the messages exactly like the d-dimensional IRS. In other words, we can expand a d-dimensional IRS to a $(d+1)$-dimensional IRS.

Lemma 5. If a graph G supports a $\langle 1, d\rangle$-MLIRS ($\langle 1, d\rangle$-MSLIRS) it also supports a $\langle 1, d+1\rangle$-MLIRS ($\langle 1, d+1\rangle$-MSLIRS).

Now, we have all the tools we need to prove the main theorem of this section.

Theorem 1. *A graph G has a $\langle 1, d\rangle$-MLIRS if and only if it is not a $(2d+1)$-windmill graph.*

Proof. First, we show that if a graph is not in the class of $(2d+1)$-windmill graphs, then it has a $\langle 1, d\rangle$-MLIRS. We use induction on d, the number of dimensions. Fraigniaud and Gavoille [FG94] have proved that if a graph G is not a lithium graph, which is exactly a 3-windmill graph, then there is a 1-LIRS for G (a $\langle 1, 1\rangle$-MLIRS). This is the basis of the induction.

Let us suppose that for any $i \leqslant d-1$, if a graph is not a $(2i+1)$-windmill graph, it has a $\langle 1, i\rangle$-MLIRS. Now, we want to show that if a graph G is not a $(2d+1)$-windmill graph, $d > 1$, then it has a $\langle 1, d\rangle$-MLIRS. We first show how to label the vertices of G. Then, we describe how we can update intervals in each step of the induction. Finally, we prove the correctness of such vertex and link labeling.

Labeling vertices:
Although G is not a $(2d+1)$-windmill graph it can be a $(2d-1)$-windmill graph. If G is not a $(2d-1)$-windmill graph, by the induction hypothesis it has a $\langle 1, d-1\rangle$-MLIRS and by Lemma 5, G also has a $\langle 1, d\rangle$-MLIRS, completing the proof. Hence, we can assume that G is a $(2d-1)$-windmill graph and by recalling Lemma 2, we can assume that G is not a chain. Therefore, by Lemma 3, G has a perfect chain, say C_1, as a proper induced subgraph. Since G is not a $(2d+1)$-windmill graph and $d > 1$, by applying Lemma 4 we can remove C_1 and the resulting graph will not be a $2d$-windmill graph. Since $2d > 3$, we can repeat these steps and remove another perfect chain, C_2, so that the resulting graph, G', is not a $(2d-1)$-windmill graph.

By the induction hypothesis, G' has a $\langle 1, d-1 \rangle$-MLIRS. We just need to expand this labeling to a $\langle 1, d \rangle$-MLIRS for G.

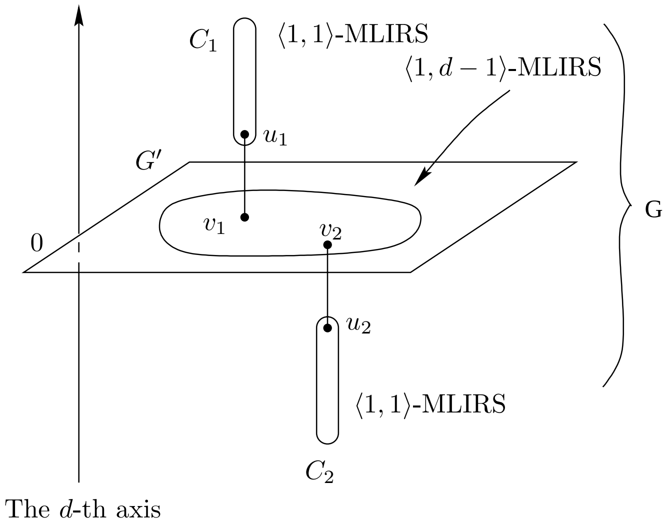

Figure 6: Expanding the labels of vertices in G' to labels for vertices in G.

C_1 and C_2 are chains and therefore, by Lemma 2, they are not 3-windmill graphs. Therefore, by the induction hypothesis, there is a $\langle 1, 1 \rangle$-MLIRS for each of them. In fact, in [FG94], it has been proved that if a given graph is not a 3-windmill (lithium) graph, we can specify a vertex and find a labeling for the vertices such that the label of the specified vertex is 1. We find such a $\langle 1, 1 \rangle$-MLIRS for C_1 (C_2) such that the label for the vertex in C_1 (C_2) joining C_1 (C_2) to the rest of the graph G, say u_1 (u_2), is 1 (Figure 6).

To construct the new labeling for G, each vertex in G' is assigned a d-dimensional label in which the first $d-1$ coordinates are the same as the labels in the linear $\langle 1, d-1 \rangle$-MIRS corresponding to G' and the dth coordinate is 0. Figure 6 illustrates an example in which $d = 3$. The third coordinates of the labels assigned to the vertices of G' are all 0, so G' lies in the plane passing through the first and the second axes. For now, we assume that the labels assigned to the vertices can have any integer value (including 0 and negative integers) as their dth coordinates. We can shift all the labels such that the dth coordinates of all labels becomes positive later.

Let (v_1, u_1) and (v_2, u_2) respectively denote the bridges connecting G' to C_1 and C_2 and let v_1 and v_2 be vertices of G'. We will set the first $d-1$ coordinates of each vertex in C_1 to be equal to the first $d-1$ coordinates of v_1. The dth coordinates of vertex labels in C_1 are the labels assigned to vertices in the previously mentioned $\langle 1, 1 \rangle$-MLIRS. In Figure 6 the vertices in C_1 all lie on the line passing through v_1 and parallel to the dth axis.

For the vertices in C_2, we will similarly set the first $d-1$ coordinates of each vertex equal to the first $d-1$ coordinates of v. If the label of a vertex v in the previously mentioned

$\langle 1,1\rangle$-MLIRS is $l(v)$, we assign $-l(v)$ as the dth coordinate of the new labeling (Figure 6). Now as mentioned before, we can shift the dth coordinate of all the labels such that the dth coordinate of the vertex with minimum value becomes 1. We let s denote the amount of this shifting and M denote the maximum value in the dth coordinate of all new labels.

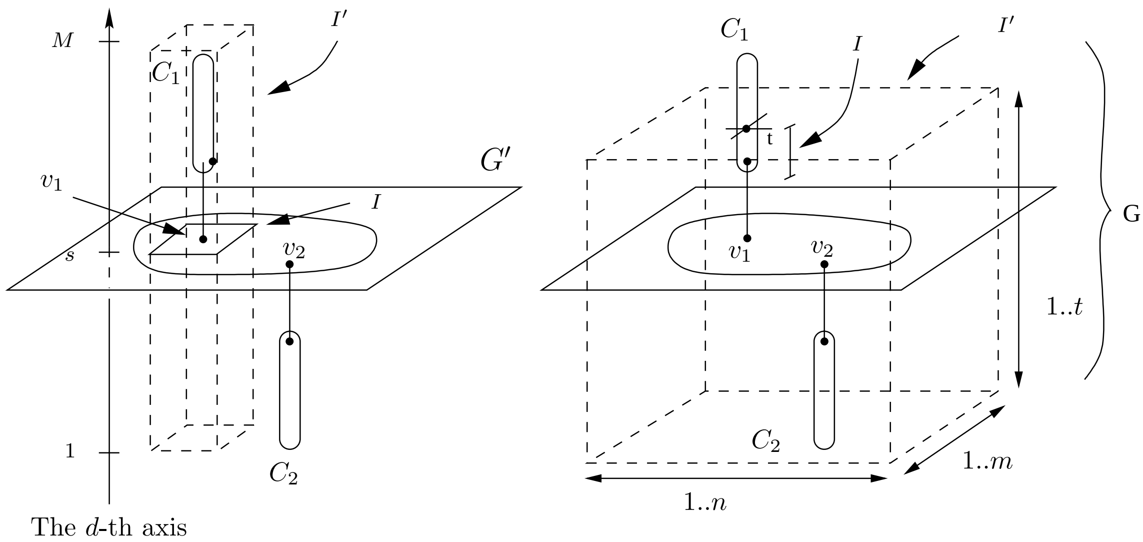

Figure 7: (a) Updating an interval in G' (b) Updating an interval, which includes u_1, in C_1 (I is the old interval, I' is the new one in both (a) and (b))

Updating Intervals:
We update intervals as follows: the first $d-1$ coordinates of each interval assigned to a link in G' is the same as the $(d-1)$-dimensional interval associated with that edge in the $\langle 1, d-1\rangle$-MLIRS defined on G'. The dth coordinate of all intervals is set to be $[1..M]$. Any $(d-1)$-dimensional interval in G' that does not contain v_1 or v_2 will still contain the same set of vertices and any interval containing v_1 (respectively v_2) will also contain all the vertices in C_1 (C_2). For example the two dimensional interval I, shown in Figure 7 (a), contains v_1, so the new three-dimensional interval I' contains all the vertices in C_1. Since I does not contain v_2, I' does not contain any of the vertices in C_2.

For the intervals associated with the links in C_1 or C_2, the first $d-1$ coordinates are set to $[1..n]$. To set the dth coordinate of each interval we will use the previously mentioned $\langle 1,1\rangle$-MLIRS. Let us assume that in the $\langle 1,1\rangle$-MLIRS defined on C_1 the interval assigned to a link e is $I_e = [a..b]$. If I_e does not contain u_1, the dth coordinate of the newly assigned d-dimensional interval will be $[a+s..b+s]$ (we shift the dth coordinate by s units because we have already shifted the vertices in this dimension). If I_e contains u_1, i.e. $I_e = [1..b]$ for some b, the dth coordinate of the newly assigned interval will be $I_e = [1..b+s]$. This means that any 1-dimensional interval defined in C_1 will be transformed into a d-dimensional interval containing the same set of vertices in C_1 and if it contains u_1, it will also contain all the vertices in G' and C_2. The interval I depicted in Figure 7 (b) contains u_1, so the new interval I' contains the set of vertices in C_1 that where in I and also all the vertices in C_2 and G'. We will analogously assign intervals to the links in C_2.

The only remaining labels to update are labels of the links $(v_1, u_1), (u_1, v_1), (v_2, u_2)$ and (u_2, v_2). The first $d-1$ coordinates of intervals associated with $(v_1, u_1), (u_1, v_1), (v_2, u_2)$ and (u_2, v_2) are set to $[1..n]$ and the dth coordinates will respectively be $[s+1..n], [1..s], [1..s-1]$ and $[s..n]$.

Correctness:
Now, let us consider a message originating from vertex w_s and with destination w_t. If both w_s and w_t are in C_1 (similarly C_2 or G') one can easily check that the newly defined $\langle 1, d \rangle$-MLIRS will route the messages on the same path as the $\langle 1, 1 \rangle$-MLIRS defined on C_1 (C_2 or the $\langle 1, d-1 \rangle$-MLIRS defined on G'). This is because if we just considering the set of vertices in C_1 (C_2 or G') each interval assigned to a link contains the same set of vertices as it contained before expanding the labels to d dimensions. If w_s is in C_1 and w_t in G', the message must go through the link (u_1, v_1) because this is the only link connecting C_1 to G'. The intervals in C_1 which contain w_t are exactly the intervals containing u_1. Therefore, this message will be forwarded through the same links as the links through which a message towards u_1 would be forwarded. When the message reaches u_1, the bridge (u_1, v_1) forwards the message to v_1, because the interval assigned to (u_1, v_1) contains all the vertices in G' and C_2. The rest of the routing will be the same as the $\langle 1, d-1 \rangle$-MLIRS defined on G'.

We can show that if there is a message in node x ($x = u_2, v_1$ or v_2) which is supposed to be forwarded the bridge connected to x, say e_x ($e_x = (u_2, v_2), (v_1, u_1)$ or (v_2, u_2) respectively), will be sent to the other end of e_x. Verifying the cases in which w_s is in C_2 or G' is similar. Hence, any message originating at any vertex and going to an arbitrary destination will eventually reach the destination, and the $\langle 1, d \rangle$-MLIRS routes messages on G properly.

We now have shown that if a graph is not in the class of $(2d+1)$-windmill graphs it has a $\langle 1, d \rangle$-MLIRS. Lemma 1 shows that no graph in this class can support a $\langle 1, d \rangle$-MLIRS. Combining these two results completes the proof of the theorem. ■

Since for each $d > 1$, we have a $(2d+1)$-windmill graph which is not a $(2d+3)$-windmill graph (for example the Y_{2d+1} graph), we can state the following corollary:

Corollary 1. *The class of graphs supporting $\langle 1, d \rangle$-MLIRS is a strict subset of the class of graphs supporting $\langle 1, d+1 \rangle$-MLIRS.*

In other words, increasing the number of dimensions increases the power of the routing scheme.

4 Characterization of networks supporting $\langle 1, d\rangle$-MSLIRS

In this section we will give a characterization of the class of graphs supporting $\langle 1, d\rangle$-MSLIRS. We will give some new definitions and will show that with slight changes in some steps in proofs, we can use the same ideas used to characterize the class of graphs supporting $\langle 1, d\rangle$-MLIRS.

In proving the Lemma 1, we needed to have at least two vertices in each arm of a $(2d+1)$-windmill graph. Otherwise, if the arm which did not have any vertex in the boundary set, say A_i, had just one vertex, say x, the interval assigned to the edge connecting A_i to R could contain x and this was not a contradiction. On the other hand, if the intervals assigned to the links are supposed to be strict, we could prove a similar lemma, even if we had an arm having just one vertex. This is the main difference between the proofs of this section and the previous one. More formally, let us start with a new definition.

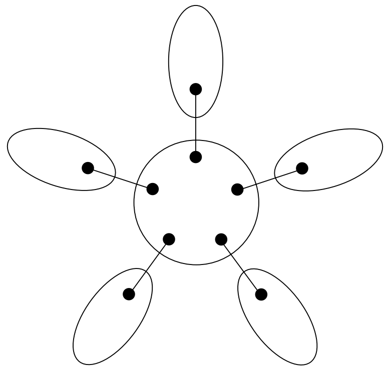

Figure 8: A weak 5-windmill graph.

Definition 4. A *weak k-windmill* graph is a connected graph G with $k+1$ connected components $A_1, A_2, ..., A_k$ (arms) and R (center) such that:

(i) there is no edge in G connecting A_i to A_j for $1 \leqslant i, j \leqslant k$ and $i \neq j$;

(ii) each component $A_i, 1 \leqslant i \leqslant k$ is connected with R by exactly one bridge (Figure 8).

As mentioned above, if the IRS is strict, then with even one vertex in each arm the proof of Lemma 1 will still be valid, because a vertex which is not in the boundary set is contained in an edge connected to it. Therefore, any weak $(2d+1)$-windmill graph does not have a $\langle 1, d\rangle$-MSLIRS. We can also verify, with the same argument as the proof of Lemma 4, that removing any perfect chain from a graph G which is not a weak k-windmill graph will produce a graph which is not a weak $(k-1)$-windmill graph.

The only remaining step is to show that the induction basis and step are also valid in constructing a $\langle 1, d \rangle$-MSLIRS for any graph that is not a weak $(2d+1)$-windmill graph. We already know that any graph which is not weak 3-windmill graph (a weak lithium graph as defined in [FG94]) has a $\langle 1, 1 \rangle$-MSLIRS, so the induction basis is true. Since we have lemmas similar to Lemmas 3 and 4 one can verify that a similar induction step still works here. This give us the complete characterization of graph supporting $\langle 1, d \rangle$-MSLIRS as follows:

Theorem 2. *A graph G has a $\langle 1, d \rangle$-MSLIRS if and only if it is not a weak $(2d+1)$-windmill graph.*

Corollary 2. *The class of graphs supporting $\langle 1, d \rangle$-MSLIRS is a strict subset of the class of graphs supporting $\langle 1, d+1 \rangle$-MSLIRS.*

5 Conclusions and open problems

In this paper we completely characterized the class of networks supporting $\langle 1, d \rangle$-MLIRS and the class of networks supporting $\langle 1, d \rangle$-MSLIRS. We showed that increasing the number of dimensions makes the routing scheme more powerful. One natural extension to this problem is to characterize the networks having a $\langle 1, d \rangle$-MLIRS or $\langle 1, d \rangle$-MSLIRS when the network has weighted links with dynamic costs. If the routing paths are supposed to be shortest paths, and we can relabel the edges after each change in the cost of links, there is a complete characterization for $\langle 1, d \rangle$-MSLIRS [Gan01a, Gan01b]. If the intervals are the same for any costs of links, the characterization problem is open even except for the 1-dimensional case [BvLT91]. There is a partial characterization for the class of networks supporting optimum LIRS in 1-dimension [NS98]. Finally, one can consider the problem of finding bounds on the length of routing paths for each of these classes.

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