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Uniquely 2-list colorable graphs ☆

Y.G. Ganjali[a,b], M. Ghebleh[a,b], H. Hajiabolhassan[a,b,*],
M. Mirzazadeh[a,b], B.S. Sadjad[a,b]

[a] *Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran*
[b] *Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran*

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Abstract

A graph is said to be uniquely list colorable, if it admits a list assignment which induces a unique list coloring. We study uniquely list colorable graphs with a restriction on the number of colors used. In this way, we generalize a theorem which characterizes uniquely 2-list colorable graphs. We introduce the uniquely list chromatic number of a graph and make a conjecture about it which is a generalization of the well-known Brooks' theorem.

1. Introduction

We consider finite, undirected simple graphs. For necessary definitions and notations we refer the reader to standard texts such as [5].

Let G be a graph, $f:V(G)\to\mathbb{N}$ be a given map, and $t\in\mathbb{N}$. An (f,t)-*list assignment* L to G is a map, which assigns to each vertex v, a set $L(v)$ of size $f(v)$ and $|\bigcup_v L(v)|=t$. By a *list coloring* for G from such L or an L-*coloring* for short, we shall mean a proper coloring c in which $c(v)$ is chosen from $L(v)$, for each vertex v. When $f(v)=k$ for all v, we simply say (k,t)-list assignment for an (f,t)-list assignment. When the parameter t is not of special interest, we say f-list (or k-list) assignment simply. Specially, if L is a (t,t)-list assignment to G, then any L-coloring is called a t-coloring for G.

In this paper, we study the concept of uniquely list coloring which was introduced by Dinitz and Martin [1] and independently by Mahdian and Mahmoodian [4]. In

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* Correspondence address: Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran.
E-mail address: hhaji@karun.ipm.ac.ir (H. Hajiabolhassan).

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[1,4] uniquely k-list colorable graphs are introduced as graphs which admit a k-list assignment which induces a unique list coloring. In the present work, we study uniquely list colorings of graphs in a more general sense.

Definition 1. Suppose that G is a graph, $f : V(G) \to \mathbb{N}$ is a map, and $t \in \mathbb{N}$. The graph G is called to be uniquely (f,t)-list colorable if there exists an (f,t)-list assignment L to G, such that G has a unique L-coloring. We call G to be uniquely f-list colorable if it is uniquely (f,t)-list colorable for some t.

If G is a uniquely (f,t)-list (resp. f-list) colorable graph and $f(v) = k$ for each $v \in V(G)$, we simply say that G is a uniquely (k,t)-list (resp. k-list) colorable graph. In [4], all uniquely 2-list colorable graphs are characterized as follows.

Theorem A (Mahdian and Mahmoodian [4]). *A graph G is not uniquely 2-list colorable, if and only if each of its blocks is either a complete graph, a complete bipartite graph, or a cycle.*

For recent advances in uniquely list colorable graphs we direct the interested reader to [3,2].

In developing computer programs for recognition of uniquely k-list colorability of graphs, it is important to restrict the number of colors as much as possible. So if G is a uniquely k-list colorable graph, the minimum number of colors which are sufficient for a k-list assignment to G with a unique list coloring, will be an important parameter for us. Uniquely list colorable graphs are related to defining sets of graph colorings as discussed in [4], and in this application also the number of colors is an important quantity.

In the next section, we show that for every uniquely 2-list colorable graph G there exists a 2-list assignment L, such that G has a unique L-coloring and there are $\max\{3, \chi(G)\}$ colors used in L.

2. Uniquely $(2,t)$-list colorable graphs

It is easy to see that for each uniquely k-list colorable graph G, and each k-list assignment L to its vertices which induces a unique list coloring, at least $k+1$ colors must be used in L, and on the other hand, since G has an L-coloring, at least $\chi(G)$ colors must be used. So the number of colors used is at least $\max\{k+1, \chi(G)\}$ colors. Throughout this section, our goal is to prove the following theorem which implies the equality in the case $k = 2$.

Theorem. *A graph G is uniquely 2-list colorable if and only if it is uniquely $(2,t)$-list colorable, where $t = \max\{3, \chi(G)\}$.*

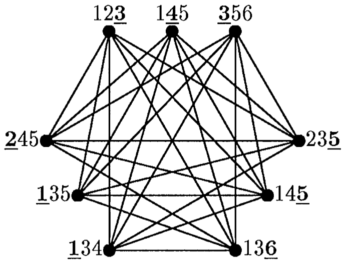

Fig. 1. A 3-list assignment to $K_{3,3,3}$ which induces a unique list coloring.

To prove the theorem above we consider a counterexample G to the statement with minimum number of vertices. In Theorems 4, 6, and 7, we will show that G is 2-connected and triangle-free, and each of its cycles is induced (chordless).

As mentioned above, if G is a uniquely k-list colorable graph, and L a (k,t)-list assignment to G such that G has a unique L-coloring, then $t \geqslant \max\{k+1, \chi(G)\}$. Although the theorem above states that when $k=2$ there exists an L for which equality holds, this is not the case in general.

To see this, consider a complete tripartite uniquely 3-list colorable graph G. We will call each of the three color classes of G a part. In [3] it is shown that for each $k \geqslant 3$ there exists a complete tripartite uniquely k-list colorable graph. For example, one can check that the graph $K_{3,3,3}$ has a unique list coloring from the lists shown in Fig. 1 (the color taken by each vertex is underlined).

Suppose that L is a $(3,t)$-list assignment to G which induces a unique list coloring c, and the vertices of a part X of G take on the same color i in c. We introduce a 2-list assignment L' to $G \setminus X$ as follows. For every vertex v in $G \setminus X$, if $i \in L(v)$ then $L'(v) = L(v) \setminus \{i\}$, and otherwise $L'(v) = L(v) \setminus \{j\}$, where $j \in L(v)$ and $j \neq c(v)$. Since L induces a unique list coloring c for G, $G \setminus X$ has exactly one L'-coloring, namely the restriction of c to $V(G) \setminus X$. But $G \setminus X$ is a complete bipartite graph and this contradicts Theorem A. So on each part of G there must appear at least 2 colors and, therefore, we have $t \geqslant 6$ while $\max\{k+1, \chi(G)\} = 4$.

Similarly, one can see that if G is a complete tripartite uniquely k-list colorable graph for some $k \geqslant 3$, and L a (k,t)-list assignment to G which induces a unique list coloring, then on each part there are at least $k-1$ colors appeared and so we have $t \geqslant 3(k-1)$ while $\max\{k+1, \chi(G)\} = k+1$.

Towards our main theorem, we start with two basic lemmas.

Lemma 2. *Suppose that G is a connected graph and $f: V(G) \to \{1,2\}$ such that $f(v_0) = 1$ for some vertex v_0 of G. Then G is a uniquely $(f, \chi(G))$-list colorable graph.*

Proof. Consider a spanning tree T in G rooted at v_0 and consider a $\chi(G)$-coloring c for G. Let $L(v)$ be $\{c(v)\}$ if $f(v) = 1$, and $\{c(u), c(v)\}$ if $f(v) = 2$, where u is the parent of v in T. It is easy to see that c is the only L-coloring of G. $\square$

Lemma 3. *Let G be the union of two graphs G_1 and G_2 which are joined in exactly one vertex v_0. Then G is uniquely $(2,t)$-list colorable if and only if at least one of G_1 and G_2 is uniquely $(2,t)$-list colorable.*

Proof. If either G_1 or G_2 is a uniquely $(2,t)$-list colorable graph, by using Lemma 2, it is obvious that G is also uniquely $(2,t)$-list colorable. On the other hand, suppose that none of G_1 and G_2 is a uniquely $(2,t)$-list colorable graph and L is a $(2,t)$-list assignment to G which induces a list coloring c. Since G_1 and G_2 are not uniquely $(2,t)$-list colorable, each of these has another coloring, say c_1 and c_2, respectively. If $c_1(v_0)=c(v_0)$ or $c_2(v_0)=c(v_0)$ then an L-coloring for G different from c is obtained obviously. Otherwise $c_1(v_0)=c_2(v_0)$, so we obtain a new L-coloring for G, by combining c_1 and c_2. □

The following theorem is immediately followed by Lemmas 2 and 3.

Theorem 4. *Suppose that G is a graph and $t \geqslant \chi(G)$. The graph G is uniquely $(2,t)$-list colorable if and only if at least one of its blocks is a uniquely $(2,t)$-list colorable graph.*

The next lemma which is an obvious statement, is useful throughout the paper.

Lemma 5. *Suppose that the independent vertices u and v in a graph G take on different colors in each t-coloring of G. Then the graph G is uniquely (f,t)-list colorable if and only if $G+uv$ is a uniquely (f,t)-list colorable graph.*

The foregoing two theorems are major steps in the proof of Theorem 11. Before we proceed, we must recall the definition of a θ-graph. If p, q, and r are positive integers and at most one of them equals 1, by $\theta_{p,q,r}$ we mean a graph which consists of three internally disjoint paths of length p, q, and r which have the same endpoints. For example, the graph $\theta_{2,2,4}$ is shown in Fig. 2.

Theorem 6. *Suppose that G is a 2-connected graph, $t=\max\{3,\chi(G)\}$, and G is not uniquely $(2,t)$-list colorable. Then G is either a complete or a triangle-free graph.*

Proof. Let G be a graph which is not uniquely $(2,t)$-list colorable for $t=\max\{3,\chi(G)\}$, and suppose that G contains a triangle. For every pair of independent vertices of G, say u and v, which take on different colors in each t-coloring of G, we add the edge uv, to obtain a graph G^*. By Lemma 5, G^* is not a uniquely $(2,t)$-list colorable graph. If G^* is not a complete graph, since it is 2-connected and contains a triangle, it must have an induced $\theta_{1,2,r}$ subgraph, say H (to see this, consider a maximum clique in G^* and a minimum path outside it which joins two vertices of this clique). Suppose that x, y, and z are the vertices of a triangle in H, and $y=v_0,v_1,\ldots,v_{r-1},v_r=z$ is a path of length r in H not passing through x. Consider a t-coloring c of G^* in which x and

v_{r-1} take on the same color. We define a 2-list assignment L to H as follows.

$$L(x) = L(z) = \{c(x), c(z)\}, L(y) = \{c(x), c(y)\},$$

$$L(v_i) = \{c(v_i), c(v_{i-1})\}, \quad \forall 1 \leqslant i \leqslant r-1.$$

In each L-coloring of H one of the vertices x and z must take on the color $c(x)$ and the other takes on the color $c(z)$. So y must take on the color $c(y)$ and one can see by induction that each v_i must take on the color $c(v_i)$, and finally x must take on the color $c(x)$. Now since G^* is connected, as in the proof of Lemma 2, one can extend L to a 2-list assignment to G^* such that c is the only L-coloring of G^*. This contradiction implies that G^* is a complete graph, and this means that G has chromatic number $n(G)$, so G must be a complete graph. □

Theorem 7. *Let G be a triangle-free 2-connected graph which contains a cycle with a chord and $t = \max\{3, \chi(G)\}$. Then G is uniquely $(2,t)$-list colorable if and only if it is not a complete bipartite graph.*

Proof. By Theorem A, a complete bipartite graph is not uniquely 2-list colorable. So if G is uniquely $(2,t)$-list colorable, it is not a complete bipartite graph. For the converse, let G be a graph which is not uniquely $(2,t)$-list colorable where $t = \max\{3, \chi(G)\}$, and suppose that G contains a cycle with a chord. For every pair of independent vertices of G, say u and v, which take on different colors in each t-coloring of G, we add the edge uv, to obtain a graph G^*. By Lemma 5, G^* is not a uniquely $(2,t)$-list colorable graph. If G^* contains a triangle, by Theorem 6, G^* and so G must be complete graphs which contradicts the hypothesis. So suppose that G^* does not contain a triangle.

Consider a cycle $v_1 v_2 \ldots v_p v_1$ with a chord $v_1 v_\ell$, and suppose H to be the graph $G^*[v_1, v_2, \ldots, v_p]$. If $v_p v_{\ell-1} \notin E(H)$, there exists a t-coloring c of G^*, such that $c(v_p) = c(v_{\ell-1})$. Assign the list $L(v_i) = \{c(v_i), c(v_{i-1})\}$ to each v_i, where $1 \leqslant i \leqslant p$ and $v_0 = v_p$. Consider an L-coloring c' for H. Starting from v_1 and considering each of two possible colors for it, we conclude that $c'(v_\ell) = c(v_\ell)$. So for each $1 \leqslant i \leqslant p$ we have $c'(v_i) = c(v_i)$. This means that H is a uniquely $(2,t)$-list colorable graph, and similar to the proof of Lemma 2, G^* is a uniquely $(2,t)$-list colorable graph, a contradiction. So $v_p v_{\ell-1} \in E(H)$ and similarly $v_2 v_{\ell+1} \in E(H)$. Now, consider the cycle $v_1 v_2 v_{\ell+1} v_\ell v_{\ell-1} v_p v_1$ with chord $v_1 v_\ell$. By a similar argument, $v_p v_{\ell+1}$ and $v_2 v_{\ell-1}$ are in $E(H)$ and so the graph $G^*[v_1 v_2 v_{\ell+1} v_l v_{\ell-1} v_p]$ is a $K_{3,3}$.

Suppose that K is a maximal complete bipartite subgraph of G^* containing the $K_{3,3}$ determined above. Since G is triangle-free, K is an induced subgraph of G. If $V(G) \setminus V(K) \neq \emptyset$, consider a vertex $v \in V(G) \setminus V(K)$ which is adjacent to a vertex w_1 of K. By 2-connectivity of G^*, there exists a path $vu_1 \ldots u_r w_2$ in which $w_2 \in V(K)$ and $u_i \notin V(K)$ for each $0 \leqslant i \leqslant r$. If w_1 and w_2 are in the same part of K, since each part of K has at least 3 vertices, there exists a vertex w_3 other than w_1 and w_2 in the same part of K as w_1 and w_2, and vertices w_1' and w_2' in the other part of K. Considering the cycle $vu_1 \ldots u_r w_2 w_2' w_3 w_1' w_1 v$ with chord $w_1 w_2'$, by a similar argument as in the previous paragraph, it is implied that v is adjacent to w_3. So v is adjacent to all the vertices of K which are in the same part of K as w_1, except possibly to w_2, but in fact v is adjacent to w_2, since we can now consider w_3 in place of w_2 and do the same as above. This contradicts the maximality of K. On the other hand if w_1 and w_2 are in different parts of K, a similar argument yields a contradiction.

We showed that $G^* = K$ and it remains only to show that $G = G^*$. If xy is an edge in G^* which is not present in G, using the fact that G is bipartite, one can easily obtain a t-coloring $(t = 3)$ of G in which x and y take on the same color, a contradiction. □

At this point, we will consider graphs that do not satisfy the conditions of Theorem 7, namely 2-connected graphs in which every cycle is induced. The following lemma helps us to treat such graphs.

Lemma 8. *A* 2*-connected graph in which each cycle is chordless, has at least a vertex of degree* 2.

Proof. It is a well-known theorem of Whitney [6] that a graph is 2-connected, if and only if it admits an ear decomposition (for a description of ear decomposition, see Theorem 4.2.7 in [5]). In the case of the present lemma, since the graph is chordless, each ear is a path of length at least 2, so the last ear contains a vertex of degree 2. □

If G is a graph and v a vertex of G, we define G_v to be a graph obtained by identifying v and all of its neighbors to a single vertex $[v]$.

Lemma 9. *If v is a vertex of degree* 2 *in a graph G, and G_v is uniquely* $(2,t)$*-list colorable for some t, then G is also uniquely* $(2,t)$*-list colorable.*

Proof. Suppose that v_1 and v_2 are the neighbors of v in G. If L is a $(2,t)$-list assignment to G_v such that G_v has a unique L-coloring, one can assign $L(w)$ to each vertex w of the graph G except v, v_1, and v_2, and $L([v])$ to these three vertices, to obtain a $(2,t)$-list assignment to G from which G has a unique list coloring. □

The following lemma gives us a family of uniquely $(2,3)$-list colorable graphs, which we will use in the proof of our main result.

Lemma 10. *Aside from $\theta_{2,2,2} = K_{2,3}$, each graph $\theta_{p,q,r}$ is uniquely* $(2,3)$*-list colorable.*

Proof. Suppose that $G = \theta_{p,q,r}$ is a counterexample with minimum number of vertices, and u and v are the two vertices of G with degree 3. If one of p, q, and r is 1, then G is a cycle with a chord and we have nothing to prove. Otherwise, suppose that one of the numbers p, q, and r, say p is odd, and there exists a vertex w on a path with length p between u and v. Then by Lemma 9, the graph G_w is not a uniquely $(2,3)$-list colorable graph, a contradiction. Hence, $p = 1$ and we yield to the previous case.

So assume that p, q, and r are all even numbers. By the hypothesis, at least one of p, q, and r, say r, is greater than 2. If either $p > 2$, $q > 2$, or $r > 4$, by use of Lemma 9,

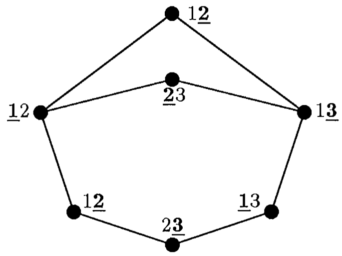

Fig. 2. The graph $\theta_{2,2,4}$.

we obtain a smaller counterexample to the statement, which is impossible by minimality of G, so $G=\theta_{2,2,4}$. In Fig. 2 there is given a $(2,3)$-list assignment to $\theta_{2,2,4}$ which induces a unique list coloring. This shows that G is a uniquely $(2,3)$-list colorable graph, which contradicts the fact that G is a counterexample to the statement. □

Now we can prove the main result.

Theorem 11 (MAIN). *A graph G is uniquely 2-list colorable if and only if it is uniquely $(2,t)$-list colorable, where $t=\max\{3,\chi(G)\}$.*

Proof. By definition, if G is uniquely $(2,t)$-list colorable for some t, it is uniquely 2-list colorable. So we must only prove that every uniquely 2-list colorable graph G is uniquely $(2,t)$-list colorable for $t=\max\{3,\chi(G)\}$. Suppose that G is a counterexample to the statement with minimum number of vertices. By Theorem 4, G is 2-connected, by Theorem 6, it is triangle-free (by Theorem A it cannot be a complete graph), and by Theorem 7, it does not have a cycle with a chord, so Lemma 8 implies that G has a vertex v with exactly two neighbors v_1 and v_2.

Consider the graph $H=G\setminus v$ and note that since $\deg v=2$, we have $\max\{3,\chi(H)\}=\max\{3,\chi(G)\}$. So if H is uniquely 2-list colorable, by minimality of G, the graph H must be uniquely $(2,t)$-list colorable, and since $t\geqslant 3$ and $\deg v=2$, we conclude that G is uniquely $(2,t)$-list colorable, a contradiction. Therefore, H is not a uniquely 2-list colorable graph and because it is a triangle-free graph, by Theorem A every block of H is either a cycle of length at least four or a complete bipartite graph. This shows that $t=3$.

We will show by case analysis that G has an induced subgraph G' which is isomorphic to some $\theta_{p,q,r}\neq\theta_{2,2,2}$ (except in case (i.2)). The graph G' is uniquely $(2,t)$-list colorable by Lemma 10. Now a $(2,3)$-list assignment to G' with a unique list coloring can simply be extended to the whole of G. This completes the proof. □

To show the existence of G' we consider two cases.

(i) The graph H is 2-connected. So H is either a K_2, a cycle, or a complete bipartite graph with at least two vertices in each part. If $H=K_2$ then $G=K_3$, a contradiction.

(i.1) If H is a cycle, G is a θ-graph and $G' = G$. Note that since G is uniquely 2-list colorable, $G' = G$ is not isomorphic to $\theta_{2,2,2}$.

(i.2) If H is a complete bipartite graph, since G is triangle-free, v_1 and v_2 are in the same part in H. Now there must exist at least one other vertex v_3 in that part – otherwise G will be a complete bipartite graph. Suppose that u_1 and u_2 are two vertices in the other part of H. The graph G' induced from G on $\{v, v_1, v_2, v_3, u_1, u_2\}$ is a uniquely $(2,3)$-list colorable with the list assignment L as follows: $L(v) = \{1,2\}$, $L(v_1) = \{1,3\}$, $L(v_2) = \{1,2\}$, $L(v_3) = \{2,3\}$, $L(u_1) = \{2,3\}$, $L(u_2) = \{1,3\}$.

(ii) The graph H is not 2-connected. Since G is 2-connected H has exactly two end-blocks each of them contains one of v_1 and v_2.
If all of the blocks of H are isomorphic to K_2, then G is a cycle which is impossible. So H has a block B with at least three vertices. Since B is a cycle or a complete bipartite graph with at least two vertices in each part, it has an induced cycle C which shares a vertex with at least two other blocks. Since G is 2-connected, these two vertices must be connected by a path disjoint from B. Suppose that P is such a path with minimum length. The graph $G' = C \cup P$ is the required θ-graph. □

3. Concluding remarks

We begin with a definition which is a natural consequence of the aforementioned results.

Definition 12. For a graph G and a positive integer k, we define $\chi_u(G,k)$ to be the minimum number t, such that G is a uniquely (k,t)-list colorable graph, and zero if G is not a uniquely k-list colorable graph. The uniquely list chromatic number of a graph G, denoted by $\chi_u(G)$, is defined to be $\max_{k \geqslant 1} \chi_u(G,k)$.

In fact, Theorem 11 states that for every uniquely 2-list colorable graph G, $\chi_u(G,2) = \max\{3, \chi(G)\}$ and by Brooks' theorem and the fact that for every uniquely 2-list colorable graph G, $\Delta(G) \geqslant 2$, we have shown that $\chi_u(G,2) \leqslant \Delta(G) + 1$. This seems to remain true if we substitute 2 by any positive integer k.

Conjecture 13. *For every graph G we have $\chi_u(G) \leqslant \Delta(G) + 1$, and equality holds if and only if G is either a complete graph or an odd cycle.*

The above conjecture implies the well-known Brooks' theorem, since for every graph G we have $\chi_u(G,1) = \chi(G)$, and so $\chi(G) \leqslant \chi_u(G)$. Hence, the above conjecture implies that $\chi(G) \leqslant \Delta(G) + 1$. On the other hand, if $\chi(G) = \Delta(G) + 1$, we will have $\chi_u(G) = \Delta(G) + 1$ and the conjecture above implies that G is either a complete graph or an odd cycle.

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References

[1] J.H. Dinitz, W.J. Martin, The stipulation polynomial of a uniquely list-colorable graph, Austral. J. Combin. 11 (1995) 105–115.
[2] Ch. Eslahchi, M. Ghebleh, H. Hajiabolhassan, Some concepts in list coloring, submitted.
[3] M. Ghebleh, E.S. Mahmoodian, On uniquely list colorable graphs, submitted.
[4] M. Mahdian, E.S. Mahmoodian, A characterization of uniquely 2-list colorable graphs, Ars Combin., to appear.
[5] D.B. West, Introduction to Graph Theory, Prentice-Hall, Upper Saddle River, NJ, 1996.
[6] H. Whitney, Congruent graphs and the connectivity of graphs, Amer. J. Math. 54 (1932) 150–168.